MOTION AND RELATIVITY

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MOTION AND RELATIVITY

by

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INTRODUCTION

The problem of motion in gravitational theory was first solved in a paper by Einstein, Infeld and Hoffmann in 1938. The calculations were so troublesome that we had to leave on reference at the Institute for Advanced Study in Princeton a whole manuscript of calculations for others to use. After that, Einstein and I made some progress together on this problem. Twenty-two years have elapsed since the first paper was published and I have again worked on this problem with my students in Warsaw during the last few years. This book presents the final results of all our work. Parallel, independently of us and a little later, W. Fock and his school in Leningrad tackled and solved the problem of motion in relativity theory, too. His results are also presented in his book entitled "Theory of space, time and gravitation" Pergamon Press, London, New York 1960. Though our approach is different from Fock's and more in the spirit of Einstein, this work is not intended to be polemic.

I have written this book with Dr Jerzy Plebański. We discussed the contents carefully over the four years it took to write it. Unfortunately we finished only the first chapter and appendix when Dr Plebański received a Rockefeller Fellowship to go to the United States. Before leaving, Dr Plebański prepared a sketch in Polish of the rest of the manuscript with the exception of the last chapter. This presentation was later very much changed by me for which I take the full responsibility.

This book presupposes only a knowledge of the general principles of relativity theory. Readers of greater mathematical inclination are advised to read the appendix first and not merely the short chapter on notation which is a summary of it.

In writing the book, we were greatly helped by Dr Andrzej Trautman who made many critical remarks, checked the formulas and prepared the bibliography. Our thanks are also due to Dr W. Tulczyjew who helped me greatly in preparing the last sections of Chapters IV and V.

Leopold Infeld

Warsaw 1960

NOTATION

A. NOTATION OF GENERAL RELATIVITY THEORY

We shall use throughout the tensor analysis of General Relativity Theory (G. R. T. for short).

We shall denote by

$$x^0, x^1, x^2, x^3 \tag{0.1}$$

the time and space coordinates of a Riemannian manifold. If we assume Special Relativity and a Cartesian coordinate system, then x^0 corresponds to the time t from

$$x^0 = ct,$$

c being the velocity of light; for $k = 1, 2, 3$, the x^k (or $\mathbf{x}$) denote the space coordinates.

All Greek indices run from 0 to 3, Latin indices from 1 to 3. Repetition of indices implies summation.

The geometry of the Riemannian space-time continuum is characterized by a symmetrical metric tensor

$$g_{\alpha\beta}(x^\mu) = g_{\beta\alpha}(x^\mu). \tag{0.2}$$

To distinguish between time and space in all possible coordinate systems we must assume that the metric tensor always satisfies the condition:

$$g_{00} > 0, \qquad \left(g_{ab} - \frac{g_{0a}g_{0b}}{g_{00}}\right) y^a y^b < 0 \tag{0.3}$$

for arbitrary $y^a \neq 0$.

Instead of these, we may assume the equivalent Hilbert conditions restricting the arbitrariness of space-time transformations:

$$g_{00} > 0, \qquad \begin{vmatrix} g_{00}, & g_{01} \\ g_{10}, & g_{11} \end{vmatrix} < 0, \qquad \begin{vmatrix} g_{00}, & g_{01}, & g_{02} \\ g_{10}, & g_{11}, & g_{12} \\ g_{20}, & g_{21}, & g_{22} \end{vmatrix} > 0, \qquad g = \det \|g_{\alpha\beta}\| < 0. \tag{0.3a}$$

The metric tensor $g_{\alpha\beta}$ is a generalization of the Minkowski metric tensor $\eta_{\alpha\beta}$ of Special Relativity Theory, defined by

$$\eta_{00} = 1, \qquad \eta_{0a} = 0, \qquad -\eta_{ab} = \delta_{ab} = \begin{cases} 1 & \text{for} \quad a = b, \\ 0 & \text{for} \quad a \neq b. \end{cases} \tag{0.4}$$

To the covariant metric tensor $g_{\alpha\beta}$ there corresponds a contravariant metric tensor $g^{\alpha\beta}$ defined by

$$g^{\alpha\varrho} g_{\varrho\beta} = \delta^\alpha_\beta = \begin{cases} 1 & \text{for} \quad \alpha = \beta, \\ 0 & \text{for} \quad \alpha \neq \beta. \end{cases} \tag{0.5}$$

We shall denote the determinant of $g_{\alpha\beta}$ by g and all quantities that transform like

$$\sqrt{-g} \times \text{tensor}$$

we shall call tensor densities and denote by

$$\mathfrak{T}^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m} = \sqrt{-g}\, T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m}. \tag{0.6}$$

The ordinary derivative will usually be denoted by a stroke:

$$S_{|\alpha} = \frac{\partial S}{\partial x^\alpha}. \tag{0.7}$$

The Christoffel symbols, which do not have tensor character, are:

$$[\alpha\beta, \gamma] = \tfrac{1}{2}(g_{\alpha\gamma|\beta} + g_{\beta\gamma|\alpha} - g_{\alpha\beta|\gamma}), \tag{0.8}$$

$$\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} = g^{\alpha\varrho}[\beta\gamma, \varrho]. \tag{0.9}$$

These symbols allow us to differentiate tensors in a covariant way. We shall denote such a covariant differentiation by a semicolon:

$$T^{\cdot\alpha\cdot\cdot}_{\cdot\cdot\cdot\cdot;\beta} = T^{\cdot\alpha\cdot\cdot}_{\cdot\cdot\cdot\cdot|\beta} + \dots + \left\{ {\alpha \atop \beta\varrho} \right\} T^{\cdot\varrho\cdot\cdot}_{\cdot\cdot\cdot\cdot} + \dots, \tag{0.10}$$

$$T^{\cdot\cdot\cdot\cdot}_{\cdot\cdot\alpha\cdot;\beta} = T^{\cdot\cdot\cdot\cdot}_{\cdot\cdot\alpha\cdot|\beta} + \dots - \left\{ {\varrho \atop \alpha\beta} \right\} T^{\cdot\cdot\cdot\cdot}_{\cdot\cdot\varrho\cdot} + \dots . \tag{0.11}$$

The indices written after the semicolon have tensor character and can be shifted up or down according to the ordinary rules.

From the Christoffel symbols we form the full Riemannian tensor:

$$R^{\mu}{}_{\nu\varrho\sigma} = \left\{ {\mu \atop \nu\sigma} \right\}_{|\varrho} - \left\{ {\mu \atop \nu\varrho} \right\}_{|\sigma} + \left\{ {\mu \atop \tau\varrho} \right\} \left\{ {\tau \atop \nu\sigma} \right\} - \left\{ {\mu \atop \tau\sigma} \right\} \left\{ {\tau \atop \nu\varrho} \right\}. \tag{0.12}$$

From it, by putting $\mu = \sigma$ we form the contracted Riemannian tensor (Ricci tensor):

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\nu\sigma} = \left\{ {\sigma \atop \mu\sigma} \right\}_{|\nu} - \left\{ {\sigma \atop \mu\nu} \right\}_{|\sigma} + \left\{ {\sigma \atop \tau\nu} \right\} \left\{ {\tau \atop \mu\sigma} \right\} - \left\{ {\sigma \atop \tau\sigma} \right\} \left\{ {\tau \atop \mu\nu} \right\}, \tag{0.13}$$

and the curvature scalar:

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{0.14}$$

The Einstein tensor is:

$$G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R. \tag{0.15}$$

All the quantities $g_{\alpha\beta}$, $\left\{ {\gamma \atop \alpha\beta} \right\}$, $R^{\mu}{}_{\nu\varrho\sigma}$, etc. are of a geometrical character. With at least some of them one associates a definite physical meaning. In a certain sense the Christoffel symbols represent the intensity of the gravitational field and the metric tensor its potentials. We shall have much more to say about this later.

B. THE δ FUNCTIONS

We shall introduce "good" δ functions — that is, δ functions which, besides having the properties of ordinary Dirac δ functions, also satisfy the condition

$$\int \frac{\delta(\mathbf{x})}{|\mathbf{x}|^p} d\mathbf{x} = 0 \quad \text{for} \quad p = 1, 2, \ldots, k. \tag{0.16}$$

All calculations in which δ functions appear and no other assumption is made, are performed with "good" δ functions. A more extensive theory of these functions is given in Appendix 1.

C. THE FIELD VALUES ON THE WORLD-LINES

Let us call $\overset{A}{\xi}{}^a(s_A)$ the world-line of the A'th singularity. Usually we take as the parameter on which the world-line depends, not the "eigentime" s_A, but the time $x^0 = ct$. Then the motion is characterized by $\overset{A}{\xi}{}^m(x^0)$ and $\xi^0 = x^0$. Let us assume

$$f = f(x^0, x^a, \overset{A}{\xi}{}^a(x^0), \overset{A}{\xi}{}^a{}_{|0}, \overset{A}{\xi}{}^a{}_{|00}, \ldots).$$

Then the "tweedling process" belonging to the A'th singularity and denoted by $\overset{A}{\sim}$ written over the "tweedled" expression means two things: first, the singular part at $x = \overset{A}{\xi}$ of f is ignored; second, it introduces the expression $\overset{A}{\xi}{}^k$ instead of x^k into the regular part of f. Or, using our "good" δ functions we define:

$$\overset{A}{\tilde{f}} = \int d\mathbf{x}\, \delta\big(\mathbf{x} - \overset{A}{\boldsymbol{\xi}}(x^0)\big) f(x^0, x^k, \overset{A}{\xi}{}^a). \tag{0.17}$$

From this definition follows (if we omit the A's):

$$\tilde{\varphi}_{|0} = \widetilde{\varphi_{|0}} + \widetilde{\varphi_{|s}}\, \xi^s{}_{|0} = \widetilde{\varphi_{|\alpha}}\, \xi^\alpha{}_{|0}, \tag{0.18}$$

$$\frac{\partial \tilde{\varphi}}{\partial \xi^s} = \widetilde{\frac{\partial \varphi}{\partial \xi^s}} + \widetilde{\varphi_{|s}}.$$

We shall assume throughout that the functions with which we deal obey the rule for "tweedling" the products, which is:

$$\widetilde{\varphi\psi} = \tilde{\varphi}\tilde{\psi}. \qquad (0\ 19)$$

For more about this process, see Appendix 2.

D. THE COVARIANT CHARACTER OF THE δ's. TENSORS ON WORLD-LINES

The four dimensional Dirac $\delta_{(4)}$ function is a scalar density. This follows from the relation:

$$\int \delta_{(4)}(x)\,dx = 1. \qquad (0.20)$$

By "world-line tensors" we shall understand tensors defined only on the world-lines ξ^a. To such tensors we may apply the rules of tensor algebra but not those of tensor analysis. To apply the latter, we must have tensor fields. We may change a tensor defined along a world-line, at least symbolically, into a tensor density field, in the following way:

$$\mathfrak{T}^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_m}(x) = \int_{-\infty}^{\infty} d\lambda\, \delta_{(4)}\bigl(x - \xi(\lambda)\bigr)\, T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_m}(\lambda). \qquad (0.21)$$

The transformation properties of $\delta_{(3)}$ follow from:

$$\delta_{(3)} = \int_{-\infty}^{\infty} \delta_{(4)}\, d\xi^0 \qquad (0.22)$$

which, together with (0.20), gives:

$$\int \delta_{(3)}\, d\mathbf{x} = 1. \qquad (0\ 23)$$

This means that $\delta_{(3)}\,d\mathbf{x}$ can be treated as an invariant. If the $\delta_{(3)}$'s are our "good" δ's, we may regard (0.22) as the definition of the "good" $\delta_{(4)}$.

For more about this, see Appendix 3.

CHAPTER I

GRAVITATIONAL INTERACTION AND THE GENERAL THEORY OF MOTION

1. GENERAL STATEMENT OF THE PHYSICAL PROBLEM. PARTICLES AND THE GRAVITATIONAL FIELD

We shall now state our problem in general terms: it is that of particles, and of the mutual gravitational interaction between them.

The particles move in a Riemannian continuum. Its geometry depends on the positions and motions of the particles; this is determined by the distribution of matter. Such is the picture that we usually have before us in G. R. T. Is it the right one? In this section we shall try to sketch an answer to this question.

First, a word about the concept of point particles. Obviously this concept is an invention of the human mind. With its help we describe the much more complicated reality. Its use implies that in a given case we may neglect the dimensions of the bodies and their structure, if we wish to describe only the essential characteristics of motion.

For this simplification we must pay a price. The picture of matter is simple, but that of the field becomes complicated. The field becomes singular on the world-lines! Yet the origin of this difficulty is of a non-physical character, therefore we are justified in treating it non-physically — that is purely formally. In doing so we shall use the mathematical tools developed in the Appendix, and sketched briefly in "Notation".

Later we shall have much more to say about the difficulties connected with the singularities of the field. They will turn out

to be important in what we are doing from the formal point of view, as our aim is to find the essential characteristic features of the motion of point particles. One should, however, not overestimate these difficulties from the physical point of view. They are not essential and they are not connected with the logical structure of G. R. T.

If we accept any coordinate system with respect to which we wish to describe the motion of point particles, then its description consists in providing

1°: The world-lines of the particles $\overset{A}{\xi}{}^{\alpha} = \overset{A}{\xi}{}^{\alpha}(\lambda)$,

2°: The metric tensor $g_{\alpha\beta}$ that characterizes the geometric properties of the continuum.

Thus, to be able to find the world-lines and the metric tensor, we must know the laws that govern the motion and the laws that govern the metric field which in turn appears to be formed by the motion.

This is a difficult problem which we shall have to approach gradually. To characterize the mechanism that connects field and motion in G. R. T., we shall start with some remarks concerning the field concept generally. At the end of this section we shall see which of these concepts can be applied to G. R. T. and which cannot.

In Newtonian physics, the concept of a field of force, and more specifically that of the gravitational field, is not of primary importance. The Newtonian gravitational theory speaks mainly about the force acting upon particles, depending only on their distances apart and their masses. This force is uniquely determined by these distances and masses, and the Newtonian equations of motion allow us, at least in principle, to find the coordinates of the point particles as functions of time. The fact that the force between particles depends essentially on their positions is usually understood in the following sense: the action of one particle upon another propagates with infinite velocity. Or, as it is often put, Newtonian Physics introduces action at a distance.

According to Relativity Theory, no action can be propagated with a velocity greater than c — the velocity of light. Thus the

field propagating from one point to another, largely because of Relativity Theory, becomes a physical reality, as real and as material as corpuscular matter. In all descriptions of nature to-day we have two principal concepts: field and matter. Besides corpuscular matter there exists the field with its laws of changes in space and time, which are of a deterministic character. These laws are as important as, if not more important than, those governing corpuscular matter.

The presumption of the existence of such real fields with a finite velocity of propagation, implies some laws concerning their structure. The fields must be described by vectors or tensors in a four-dimensional manifold as functions of x^{α}, satisfying partial differential equations of the hyperbolic type, so that Cauchy's problem can be correctly set. This is the mathematical expression of the physical statement that the field is governed by deterministic laws. Also we demand of these equations that they express the physical fact that the field is caused by moving particles. Thus, in our mathematical description, on the right-hand sides of these equations some expressions characterizing the sources of the field usually appear. It is obvious that Dirac's δ functions are a convenient mathematical tool for the charecterization of the sources and their motion if point particles are assumed.

This is the dualistic view which we usually accept in physical theories, for example Maxwell's theory. That is, we assume the existence of both the field and its sources. But what about the motion of these sources? Let us state: no linear field theory can determine the motion of its sources except in trivial cases. This can be seen in the following way. Let us assume that the motion of a particle A is determined by the field equations. We now take another solution in which the motion of another particle B is also determined by the field equations. Then, because the equations are linear, there exists a solution in which both particles A and B move exactly as before — that is, as in the solutions in which only A, or only B, moved. Thus we arrive at an absurd conclusion: the motion of A is determined by the field equations but it is not changed by the presence, or motion, of B. Therefore, from the linear theories

motion can only be deduced in the trivial case of non-interacting bodies.

Thus in every linear field theory of interacting particles we must add to the field equations the equations of motion of the sources of the field. In the case of point sources, these are ordinary differential equations of the second order, in which the acceleration of a particle is determined by the value of the field at the point where the particle is.

Thus we say: the laws of propagation of a field are usually described by linear partial differential equations of hyperbolic type; the motions of its sources, that is, of the particles or charges, are described by ordinary differential equations in which the accelerations are determined by field functions at the points where the particles are at a given moment. It is this situation which we have in mind when speaking about close-range action.

It would seem that all linear and non-linear relativistic theories should be based on close-range action in the sense just described. Yet this is not quite true. In particular we shall have to change slightly the meaning of close-range action so that it will fit Einstein's theory of gravitation.

Einstein's gravitational equations are hyperbolic differential equations for the metric field $g_{\alpha\beta}$. The components of the metric field are also understood as the potentials of the gravitational field. The sources of the field form the tensor of energy-momentum; they appear on the right-hand side of the field equations and characterize the distribution and motion of corpuscular matter. Therefore, such a situation would seem typical, fitting perfectly the scheme described before, and very similar to the case of Maxwell's equations. But there is one very essential difference. Whereas in the situation described up to now, the equations of motion have to be added to the field theory (e. g. the Lorentz equations of motion) this is not true for G. R. T. It is a non-linear theory in which the motions of the sources are determined by the equations of the field. Our aim in this section is to investigate provisionally just this peculiar situation.

For the sake of greater clarity, we shall try to say more precisely what we understand by the words "equations of motion" in gravitational theory or, for that matter, in any field theory. It will be convenient for our purpose to distinguish among equations of motion of three kinds.

1°. The equations of motion of the first kind are the equations of motion for a test particle; we shall understand by them the equations of motion of a particle on which a given external field acts, while the field does not depend on the motion of the particle itself.

By a test particle we mean, therefore, a particle that can be ignored as a source of the field; this is so when the coupling of the particle with the field is infinitely small, so that we may neglect the modification of the field caused by the presence of this particle. In the case of a gravitational field we mean by a test particle one with infinitely small mass.

According to the principle of close-range action, such an external field acts upon the test particle; the second derivatives of its coordinates with respect to the proper time are determined by the external gravitational field. Since the particle itself does not participate in the formation of the field, the field is regular along the world line of the particle. It suffices to introduce into the external field the coordinates of the test particle.

In the early days of G. R. T., these equations of motion for a test particle were postulated as the geodesic equations derived from a variational principle. Thus in G. R. T. (in the case of a test particle), as in any linear dualistic theory, we had two independent sets of equations: those of the field and those for the motion of this test particle. Both the field equations and the equations of motion can generally (in a gravitational theory as well as — for example — in Lorentz's theory) be deduced from one variational principle:

$$\delta(W_f + W_I) = 0, \tag{1.1.1}$$

where W_f depends only on the field and W_I on both the field and the world lines of the particles. In the case of a test particle we may,

however, split this equation into two independent variational principles

$$\delta W_f + \delta W_I = 0, \qquad \delta W_I = 0. \tag{1.1.1a}$$

In the first we vary the field, in the second — the world-line. Essentially all that was known in the early days of G. R. T., as far as motion was concerned, were the equations of motion for a test particle which were independently assumed.

This situation lasted until 1927, when a paper by Einstein and Grommer appeared. In this paper, it was claimed that the equations of motion for a test particle do not need to be added to the field equations, that they are a consequence of the field equations. One of the reasons that it is possible to deduce the equations of motion for a test particle from the field equations alone is that the field equations are non-linear. Thus in the theory of G. R. T., we do not need to postulate any equations of motion for a test particle. The field equations are sufficient!

2°. By equations of motion of the second kind we shall mean equations of motion for finite masses whose accelerations are functions of the field on the world-lines of the particles. Thus the field at the particle may depend on the mass of the particle itself and on its motion. But in the case of a point particle the self-field is infinite on the world-line of the particle. Therefore, we must find a definite meaning for the concept of the field at the particle. It is here that the tweedling process described in "Notation" and Appendix 2, will play an essential role.

On the other hand, we know that the field remains undetermined until the motion is determined. Therefore, the equations of motion of the second kind will have only a symbolic meaning, since they will contain an unknown quantity: the field at the point of the particle. To make these equations of motion definite, we obviously do not need to know all the field components everywhere. It is sufficient to know only the combination in which they appear in the equations of motion and to know them only on the world-line of the particle.

These general equations of the second kind can be found in practice in any field theory from a variational principle:

$$\delta W_I = 0 \tag{1.1.2}$$

in which W_I is essentially the same as before in (1.1.1a) with the sole difference that the field at the points on the world-line is not assumed to be given but to be deduced from the field equations.

We shall show later that in the case of G. R. T. we can obtain the same equations of motion that we deduce from a variational principle just as well from the field equations. The practical value of these equations is, however, small since the field which enters into them remains unknown; it must be deduced from the field equations.

3°. The equations of motion of the third kind are found from the equations of motion of the second kind by introducing into them the actual values of the field.

How is it to be done? The answer to this question is not simple. We shall try now only to indicate it in general terms and shall do it in detail later.

Let us go back to the case of a linear dualistic field theory. Let us assume that we have the field equations for arbitrary motion. In other words, we assume the case in which the field equations do not determine motion and in which the equations of motion are independently added to the field equations. It is in this case that we can solve the field equations for arbitrary motion. The field tensors — solutions of the field equations denoted symbolically by f — depend on position and functionally on the world-lines $\overset{A}{\xi}{}^{\alpha}$. Thus we can write symbolically:

$$f = f(x^{\alpha})[\overset{1}{\xi}{}^{\mu}, \overset{2}{\xi}{}^{\mu}, \ldots, \overset{N}{\xi}{}^{\mu}]. \tag{1.1.3}$$

Then, putting into f in place of x^{α}, say, $\overset{A}{\xi}{}^{\alpha}(x^0)$ we find the field at the A'th particle. Introducing these values for the field into the added equations of motion, we obtain a set of differential-functional equations which we shall call the equations of motion of the third

kind – for the world-lines of the N particles. Thus in these equations the field is already eliminated and what remains are equations formed only of the ξ's. Their integration would give us the motion of N particles.

Generally speaking, such is the situation in every linear theory in which the motion of the suces is arbitrary. But we are mostly interested in the equations of motion in G. R. T. In Einstein's gravitational theory the situation is in some respects more complicated, in others simpler, than that sketched above.

In what respect is the situation more complicated?

In G. R. T. we do not have field equations with arbitrary motion and then, additionally, the equations of motion. The same set of equations that determines the field, also determines the motion. How, then, can we find the solutions of such equations, solutions giving at the same time the field and the motion? The field is determined by the motion and the motion is determined by the field. What is the way out of this difficulty? We shall see later that there are two ways out. We shall sketch them only very briefly and simplify their description (for the moment) to such an extent as to make them not quite truthful.

The first way is: change Einstein's gravitational equations into an artificial system of different equations so chosen that it permits arbitrary motion. (This is done by the help of a non-physical field, which we shall call the dipole field). Then find the field corresponding to these non Einsteinian equations. However, in the end we wish to return to Einstein's equations. This can be done only by removing the non-physical field, and so restricting the motion to that prescribed by Einstein's equations. Thus the procedure consists in finding the field for the non-Einsteinian equations and then, at the same time, restricting both the field and the motion so that both satisfy the equations of G. R. T.

The second way is based on the approximation procedure. Roughly speaking this can be described in the following way. We start with the Newtonian field which, in the first approximation, satisfies the field equations of G. R. T. Then we find the motion belonging to this field and determined by the field equations. It is

the Newtonian motion. Then, having done this, we can find the field in the next approximation. To this field there belongs a motion calculated (with the help of this field) to the post-Newtonian approximation. At each approximation level the existence of the equations of motion forms a condition that makes it possible to carry the approximation procedure for the field one step further. In this way, step by step, we can determine field and motion with increasing accuracy.

These are the ideas that we shall employ in our study of the problem of motion in G. R. T. They make the case of the gravitational field more complicated than, say, that of the electromagnetic field. However, as we mentioned before, in one respect the situation is simpler than that for an electromagnetic or for some other linear field.

In what respect is the situation simpler?

We have said before that the equations of motion of the third kind should be (generally speaking) differential-functional equations. Or, in other words, differential equations of an infinite order, since to describe the ξ's at an arbitrary moment, all derivatives of the ξ's at that moment are necessary. Yet the gravitational equations of motion, at least up to the post-Newtonian approximation, are a set of $3N$ ordinary differential equations of the order 2. Therefore, their structure is much simpler than we could have predicted from the general case.

How does this situation in G. R. T. fit our concept of close-range action? This concept was formed for linear theories in which the motion was not determined by the field equations. Its mathematical expression consisted of the hyperbolic field equations and the differential-functional equations for the motion of the third kind.

In G. R. T. we also have hyperbolic equations for the field, as in the linear theories. However, after using the approximation method, the equations of motion that follow from the field equations turn out to be ordinary differential equations of the second order; therefore they have a form similar to the Newtonian theory of motion, that is, as in the case of action at a distance. In this respect

the concept of close-range action formed on the basis of linear field theories does not fit G. R. T.

Can these equations of the third kind be gained from a variational principle in which only the world-lines and not the field appear? We shall show that this can, indeed, be done for the equations of motion up to the post-Newtonian order.

We shall call such variational principles leading to the equations of motion of the third kind those of Fokker's type.

Summarizing the essential point briefly: It is the field which is of primary importance in G. R. T. The field equations determine not only the field; they determine the motion as well.

2. NEWTON'S THEORY OF GRAVITATION

Newton's theory of gravitation is a theory based on the concept of action at a distance. In it the gravitational field appears only as an auxiliary mathematical tool. Because of its simplicity, this theory is well suited to be an example illustrating the three kinds of equations of motion and the simple variation principles that lead to them.

Let $\overset{A}{\xi}(t)$ be the world lines of N point particles ($A = 1, 2, \ldots, N$), $\overset{A}{m}$ their masses and $\varphi(\mathbf{x}, t)$ the Newtonian gravitational field. We shall assume, as the foundation of Newton's theory, the variational principle:

$$\delta W[\varphi, \overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{N}{\xi}] = 0, \tag{1.2.1}$$

where the action W is a functional of the field $\varphi(\mathbf{x}, t)$ and the world-lines $\overset{A}{\xi}(t)$. The variation has to be performed with respect to both φ and the ξ's. The action W consists of two parts: W_f belonging to the field and W_I belonging both to the world-lines of the particles and to the field, that is, to the field-matter interaction:

$$W = W_f + W_I. \tag{1.2.2}$$

We define W_f and W_I:

$$W_f = -\frac{1}{4\pi k}\int d\mathbf{x}\int_{t_1}^{t_2} dt\,\tfrac{1}{2}\varphi_{|a}\varphi_{|a}, \tag{1.2.3}$$

k = gravitational constant)

$$W_I = \sum_{A=1}^{N}\int_{t_1}^{t_2} dt\,\tfrac{1}{2}\overset{A}{m}\,\dot{\overset{A}{\xi}}{}^{2} - \sum_{A=1}^{N}\int_{t_1}^{t_2} dt\int d\mathbf{x}\,\overset{A}{m}\,\delta(\mathbf{x}-\overset{A}{\xi}(t))\,\varphi(\mathbf{x},t). \tag{1.2.4}$$

Let us start with the field equations. They are obtained by varying $W = W_f + W_I$ with respect to φ. We have

$$\delta W = -\sum_{A=1}^{N}\int_{t_1}^{t_2} dt\int d\mathbf{x}\,\overset{A}{m}\,\delta(\mathbf{x}-\overset{A}{\xi})\,\delta\varphi(\mathbf{x},t) + $$

$$+\frac{1}{4\pi k}\int_{t_1}^{t_2} dt\int d\mathbf{x}\,\varphi_{|aa}\,\delta\varphi, \tag{1.2.5}$$

and from it:

$$\varphi_{|aa} = 4\pi k\sum_{A=1}^{N}\overset{A}{m}\,\delta(\mathbf{x}-\overset{A}{\xi}). \tag{1.2.6}$$

If we admit only solutions that vanish at infinity, then the solution of the above equation is unique:

$$\varphi = -k\sum_{A=1}^{N}\frac{\overset{A}{m}}{|\mathbf{x}-\overset{A}{\xi}|}. \tag{1.2.7}$$

Thus we can solve the Newtonian field equations with arbitrary motion, that is with arbitrary functions $\xi(t)$. The field does not depend functionally on the world-lines. The field φ is simply a function of the position coordinates of the particles, all taken at the same moment of time.

Now, as to the Newtonian equations of motion:

The equations of motion of the first kind, or, the equations of motion for a test particle in a given gravitational field, are obtained by putting in W_I $\overset{1}{m} = \Delta m$ and $\overset{A}{m} = 0$ for all the other particles

(that is, for $A = 2, 3, \ldots, N$). Assuming that φ is regular along the world-line $\xi(t)$ of the test particle, we take into account only the following part of W_I:

$$W_I' = \int_{t_1}^{t_2} dt\left(\tfrac{1}{2}\Delta m \dot{\xi}^s \dot{\xi}^s - \Delta m \tilde{\varphi}(\xi, t)\right). \tag{1.2.8}$$

Varying W_I' with respect to ξ^s, we have:

$$\delta W_I' = -\int_{t_1}^{t_2} dt\, \Delta m\, \delta\xi^s(\ddot{\xi}^s + \tilde{\varphi}_{|s}), \qquad \left(\tilde{\varphi}_{|s} = \frac{\partial\tilde{\varphi}}{\partial\xi^s}\right) \tag{1.2.9}$$

from which we obtain, as the equations of motion for a test particle in a given gravitational field φ:

$$\Delta m\, \ddot{\xi}^s = -\Delta m \tilde{\varphi}_{|s}, \tag{1.2.10}$$

and, dividing by Δm, we have, finally:

$$\ddot{\xi}^s = -\tilde{\varphi}_{|s}. \tag{1.2.11}$$

The equations of motion of the second kind we obtain by varying the full expression (1.2.4) for W_I with respect to $\overset{A}{\xi}(t)$:

$$\delta W_I = -\sum_{A=1}^{N} \int_{t_1}^{t_2} dt\, \overset{A}{m} \cdot \delta \overset{A}{\xi^s}\left(\overset{A}{\ddot{\xi}^s} - \int d\mathbf{x}\, \delta(\mathbf{x} - \overset{A}{\xi})_{|s}\varphi(\mathbf{x}, t)\right). \tag{1.2.12}$$

Integrating by parts under the integral, we obtain, because of $\partial/\partial\mathbf{x} = -\partial/\partial\xi$, the equations of motion of the second kind:

$$\overset{A}{\ddot{\xi}^s} = -\overset{A}{\widetilde{\varphi_{|s}}} = -\int d\mathbf{x}\, \delta(\mathbf{x} - \overset{A}{\xi})\varphi_{|s}. \tag{1.2.13}$$

In these equations there appear the field intensities along the world-line. But φ may be singular along a world-line. We get rid of this singularity by using our good δ functions. In the case of a test particle it was immaterial whether we used our good functions or Dirac's δ functions, since there their only purpose was to introduce the ξ in place of the $\mathbf{x}$. In the case of a test particle it was also immaterial whether we used $\tilde{\varphi}_{|s}$ or $\widetilde{\varphi_{|s}}$, since there φ did not depend on ξ. This is not immaterial when varying (1.2.12). In passing from (1.2.12) to (1.2.13) we must assume that φ, at least in its regular

part, does not depend on ξ explicitly, or, if it does, that we vary only the δ's and not the φ's with respect to the ξ's.

The equations of the third kind are gained by substituting into the equations of the second kind the field variables as functions of the world-lines of the particles; this means that we have to introduce into equation (1.2.13) for φ the value (1.2.7). We have:

$$\varphi_{|a} = k \sum_{B=1}^{N} \overset{B}{m} \frac{x^a - \overset{B}{\xi}{}^a(t)}{\left|\mathbf{x} - \overset{B}{\xi}(t)\right|^3} \tag{1.2.14}$$

so that

$$\overset{A}{\overline{\varphi_{|a}}} = k \int d\mathbf{x}\, \delta(\mathbf{x} - \overset{A}{\xi}) \sum_{B=1}^{N} \overset{B}{m} \frac{x^a - \overset{B}{\xi}{}^a}{\left|\mathbf{x} - \overset{B}{\xi}\right|^3} = k \sum_{\substack{B=1\\B\neq A}}^{N} \overset{B}{m} \frac{\overset{A}{\xi}{}^a - \overset{B}{\xi}{}^a}{\left|\overset{A}{\xi} - \overset{B}{\xi}\right|^3}. \tag{1.2.15}$$

This equation is true, because $\delta(\mathbf{x} - \overset{A}{\xi})$ replaces $\mathbf{x}$ by $\overset{A}{\xi}(t)$ in the regular part of the function. We may remark, in passing, that this equation would have been true had we used, not our good δ functions, but ordinary Dirac δ functions. This would have been so because the integral of the singular part of $\varphi_{|a}$ would have vanished on account of the spherical symmetry of the ordinary δ functions. Therefore, to obtain a regular expression for the field on the world-lines in Newton's theory, it is sufficient to appeal to the spherical symmetry of the ordinary δ function. We do not need to introduce our good δ function. Thus, introducing (1.2.15) into (1.2.13), we obtain equations of the third kind:

$$\ddot{\overset{A}{\xi}}{}^a = -k \sum_{\substack{B=1\\B\neq A}}^{N} \overset{B}{m} \frac{\overset{A}{\xi}{}^a - \overset{B}{\xi}{}^a}{\left|\overset{A}{\xi} - \overset{B}{\xi}\right|^3}, \tag{1.2.16}$$

or multiplying by $\overset{A}{m}$, the more conventional form:

$$\overset{A}{m}\, \ddot{\overset{A}{\xi}}{}^a = -k \sum_{\substack{B=1\\B\neq A}}^{N} \overset{A}{m}\overset{B}{m} \frac{\overset{A}{\xi}{}^a - \overset{B}{\xi}{}^a}{\left|\overset{A}{\xi} - \overset{B}{\xi}\right|^3}. \tag{1.2.17}$$

Now we can easily give Fokker's variational principle leading to (1.2.17). This is the variational principle, well-known from theoretical mechanics:

$$\delta W_F = \delta\left[\int_{t_1}^{t_2} dt\left(\sum_{A=1}^{N} \tfrac{1}{2}\overset{A}{m}\,\dot{\overset{A}{\xi}}{}^{s}\,\dot{\overset{A}{\xi}}{}^{s} + \tfrac{1}{2}\sum_{\substack{A,B=1\\ A\neq B}}^{N} k\,\frac{\overset{A}{m}\overset{B}{m}}{\left|\overset{A}{\xi}-\overset{B}{\xi}\right|}\right)\right] = 0, \qquad (1.2.18)$$

where the variations have to be performed with respect to the world-lines.

What is the connection between this variational principle and that for the equations of the second kind? The answer is: If into $W = W_f + W_I$ we introduce as the field φ the concrete solution

$$\varphi = -k\sum_{A=1}^{N}\frac{\overset{A}{m}}{\left|\mathbf{x}-\overset{A}{\xi}\right|} \qquad (1.2.19)$$

of the field equations, then we obtain the right expression for W_F.

To prove this theorem we start with W_f. We have the following equality which we introduce only for the sake of variation with respect to φ:

$$W_f = -\frac{1}{4\pi k}\int d\mathbf{x}\int_{t_1}^{t_2} dt\,\tfrac{1}{2}\varphi_{|a}\varphi_{a|} = \frac{1}{8\pi k}\int_{t_1}^{t_2} dt\int d\mathbf{x}\,\varphi_{|aa}\varphi. \qquad (1.2.20)$$

Substituting for φ from (1.2.19) and making use of the field equations (1.2.6), we have:

$$W_f = -\tfrac{1}{2}\sum_{A=1}^{N}\sum_{B=1}^{N} k\overset{A}{m}\overset{B}{m}\int_{t_1}^{t_2} dt\int d\mathbf{x}\,\delta(\mathbf{x}-\overset{A}{\xi})\,|\mathbf{x}-\overset{B}{\xi}|^{-1}$$

$$= -\tfrac{1}{2}\sum_{A\neq B=1}^{N}\int_{t_1}^{t_2} dt\,k\,\frac{\overset{A}{m}\overset{B}{m}}{\left|\overset{A}{\xi}-\overset{B}{\xi}\right|} - \tfrac{1}{2}\sum_{A=1}^{N}\int_{t_1}^{t_2} dt\,k\overset{A}{m}{}^{2}\int d\mathbf{x}\,\frac{\delta(\mathbf{x}-\overset{A}{\xi})}{\left|\mathbf{x}-\overset{A}{\xi}\right|}. \qquad (1.2.21)$$

The last integral vanishes, because our δ's are the good δ's. Now substituting φ from (1.2.19) into W_I, we have:

$$W_I = \int_{t_1}^{t_2} dt \sum_{A=1}^{N} \tfrac{1}{2} \overset{A}{m} \dot{\overset{A}{\xi}}{}^{s} \dot{\overset{A}{\xi}}{}^{s} + \int_{t_1}^{t_2} dt \sum_{A,B=1}^{N} k \overset{A}{m} \int d\mathbf{x}\, \delta(\mathbf{x} - \overset{A}{\xi}) \frac{\overset{B}{m}}{\left|\mathbf{x} - \overset{B}{\xi}\right|} \tag{1.2.22}$$

$$= \int_{t_1}^{t_2} dt \sum_{A=1}^{N} \tfrac{1}{2} \overset{A}{m} \dot{\overset{A}{\xi}}{}^{s} \dot{\overset{A}{\xi}}{}^{s} + \int_{t_1}^{t_2} dt \sum_{\substack{A,B=1 \\ A \neq B}}^{N} k \frac{\overset{A}{m}\overset{B}{m}}{\left|\overset{A}{\xi} - \overset{B}{\xi}\right|} + \sum_{A=1}^{N} \int_{t_1}^{t_2} dt\, k \overset{A}{m}{}^{2} \int d\mathbf{x} \frac{\delta(\mathbf{x} - \overset{A}{\xi})}{\left|\mathbf{x} - \overset{A}{\xi}\right|}.$$

Again the last integral vanishes for the same reasons as before. Adding the last two equations together we find:

$$W_f + W_I = \int_{t}^{t_2} dt \left(\sum_{A=1}^{N} \tfrac{1}{2} \overset{A}{m} \dot{\overset{A}{\xi}}{}^{s} \dot{\overset{A}{\xi}}{}^{s} + \tfrac{1}{2} \sum_{\substack{A,B=1 \\ A \neq B}}^{N} k \frac{\overset{A}{m}\overset{B}{m}}{\left|\overset{A}{\xi} - \overset{B}{\xi}\right|} \right) \tag{1.2.23}$$

which is identical with Fokker's W_F which we quoted in (1.2.18). But this complete agreement is due to the use of our good δ's. Using ordinary δ's we should have found in (1.2.23) an additional expression of the form:

$$W_{\text{self}} = \tfrac{1}{2} \int_{t_1}^{t_2} dt \sum_{A=1}^{N} k \overset{A}{m}{}^{2} \int d\mathbf{x} \frac{\delta(\mathbf{x})}{|\mathbf{x}|} \tag{1.2.24}$$

which is infinite in the case of Dirac's functions, zero in the case of our good functions, and generally could take any pre-assigned value, since we can find models of the δ functions for which

$$\int d\mathbf{x} \frac{\delta(\mathbf{x})}{|\mathbf{x}|} = \omega_{(1)}$$

where $\omega_{(1)}$ is an arbitrary pre-assigned number. In any case this additional expression would be constant and would have no influence upon the variation. Thus, as far as the variation of W_F is concerned, our statement is independent of the choice of the δ's.

The Newtonian theory of gravitation as summarized in this section uses the idealized concept of point particles. With its help we represent the essential characteristic feature of planetary motion on which the dimensions of planets and their rotation have a small influence.

G. R. T. is essentially a generalization and a broadening of the simple Newtonian theory. There too, the introduction of point particles and the use of the δ functions will allow us to describe the important features of their motion.

The Newtonian theory has been developed in this section in such a way as to allow us later to stress the differences and, of course, also the similarities between it and G. R. T. The most essential difference consists in the fact that in G. R. T. the equations of motion follow from the field equations. Yet it is the Newtonian theory which will serve us as a guide to G. R. T. Indeed, we shall look for solutions which for $c \to \infty$ go over into solutions of Newtonian theory. The principle of correspondence is just as important in the relation between G. R. T. and Newtonian theory, as it is between quantum theory and classical mechanics.

3. INTERACTION IN G. R. T. THE EQUATIONS OF MOTION OF THE FIRST AND SECOND KIND

We have N point particles and their world-lines $\overset{A}{\xi}{}^{\alpha} = \overset{A}{\xi}{}^{\alpha}(\lambda)$. We denote their masses by $\overset{A}{m}_{(0)}$. Formally these may be regarded as the coupling constants of the particles with the gravitational field.

For the interaction W_I between the particles and the gravitational field, we shall assume the following expression:

$$W_I = -\sum_{A=1}^{N} \overset{A}{m}_{(0)} c \int_{\sigma_1}^{\sigma_2} (\overset{A}{\overline{g_{\alpha\beta}}} d\overset{A}{\xi}{}^{\alpha} d\overset{A}{\xi}{}^{\beta})^{1/2}. \tag{1.3.1}$$

We shall interpret $\overset{A}{\overline{g_{\alpha\beta}}}$ according to what was said in "Notation",

or, more fully in Appendix 2, that is: put in $\overset{A}{\xi}{}^k$ instead of x^k, ignoring the singular part of $g_{\alpha\beta}$. We can write the covariant definition:

$$\overline{\overset{A}{g}_{\alpha\beta}} = \int dx\, \delta_{(4)}\big(x - \overset{A}{\xi}(\overset{A}{\lambda})\big)\, g_{\alpha\beta}(x) \tag{1.3.2}$$

where $\delta_{(4)}$ is defined by (0.22), with the help of our good δ's.

If $g_{\alpha\beta}$ is regular in x^0, as is always the case, we may, again making use of (0.22) write the above equation in the form

$$\overline{\overset{A}{g}_{\alpha\beta}} = \int d\mathbf{x}\, \delta(\mathbf{x} - \overset{A}{\xi})\, g_{\alpha\beta}(\mathbf{x}, \xi^0), \tag{1.3.3}$$

where the δ is our good δ function. We shall therefore assume that all integrals of the kind

$$\int d\mathbf{x} \frac{\delta(\mathbf{x})}{|\mathbf{x}|^p}$$

for an arbitrary given p vanish. Finally, going back to the notation (1.3.1), we may remark that σ_1 and σ_2 denote two arbitrary hypersurfaces which are space-like, that is, such that the normals to them n^α satisfy the condition $g_{\alpha\beta} n^\alpha n^\beta > 0$. The integrals must be taken along the world-lines between these two surfaces.

The expression W_I defined in such a way (if we ignore for the moment subtleties arising from the definition of field components along the world-lines) is the natural generalization of the corresponding one in Special Relativity Theory. Indeed, in the case of Minkowski space-time, that is, in the case $g_{\alpha\beta} = \eta_{\alpha\beta}$, according to (0.4) our equation (1.3.1) for W_I takes the form:

$$W_I = -\sum_{A=1}^{N} \overset{A}{m}_{(0)} c \int_{\sigma_1}^{\sigma_2} \big((d\overset{A}{\xi}{}^0)^2 - d\overset{A}{\xi}{}^a d\overset{A}{\xi}{}^a\big)^{1/2}, \tag{1.3.4}$$

that is, the well-known expression for inertial action in Special Relativity Theory. We see that our original expression (1.3.1) for W_I is the only sensible generalization of the above expression to the case of a curved world.

Thus, having W_I, we can easily obtain the differential equations of motion of the first and second kind.

We start with the equations of motion of the first kind, that is, the equations of motion for a test particle. To do this, we have to vary in (1.3.1) only the expression with mass m, for which we assume a small value Δm. We assume, too, that Δm does not influence the given field, which remains regular along the world-line of the particle. Thus the $\delta_{(4)}$ in (1.3.2) does only one thing: it introduces the world-line, that is, ξ^a in place of x^a. Therefore W_I for a test particle is, as far as its variation is concerned,

$$W_I = -\Delta m c \int_{\sigma_1}^{\sigma_2} \left(g_{\alpha\beta}(\xi) d\xi^\alpha d\xi^\beta\right)^{1/2}. \tag{1.3.5}$$

We obtain the equations of motion in the usual way by varying W_I with respect to ξ^a. Writing

$$ds = (g_{\alpha\beta} d\xi^\alpha d\xi^\beta)^{1/2}$$

we have:

$$\delta W_I = -\Delta m c\, g_{\alpha\beta} \frac{d\xi^\beta}{ds} \delta\xi^\alpha \Big|_{\sigma_1}^{\sigma_2} + \Delta m c \int_{\sigma_1}^{\sigma_2} ds\, \delta\xi^\alpha \left[\frac{d}{ds} g_{\alpha\beta} \frac{d\xi^\beta}{ds} + \right.$$
$$\left. -\tfrac{1}{2} g_{\mu\nu|\alpha} \frac{d\xi^\mu}{ds} \frac{d\xi^\nu}{ds}\right]. \tag{1.3.6}$$

If $\delta\xi^\alpha$ vanishes at σ_1 and σ_2, we obtain from the condition $\delta W_I = 0$, the equations for a test particle

$$\frac{d}{ds}\left(g_{\alpha\beta} \frac{d\xi^\beta}{ds}\right) - \tfrac{1}{2} g_{\mu\nu|\alpha} \frac{d\xi^\mu}{ds} \frac{d\xi^\nu}{ds} = 0 \tag{1.3.7}$$

where, of course, $g_{\alpha\beta}$ and its derivatives have to be taken at the point ξ^α. Performing the differentiation $\dfrac{d}{ds}\left(g_{\alpha\beta}\dfrac{d\xi^\beta}{ds}\right)$ in the last equation, and raising the index α by multiplying this equation by

$g^{\alpha\varrho}$, we obtain:

$$\frac{d^2 \xi^\varrho}{ds^2} + \begin{Bmatrix} \varrho \\ \mu\nu \end{Bmatrix} \frac{d\xi^\mu}{ds} \frac{d\xi^\nu}{ds} = 0 \tag{1.3.8}$$

where the Christoffel symbol has to be taken at the point ξ^ν. Using the symbol D for covariant differentiation, we have:

$$DT^\alpha = dT^\alpha + \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} T^\mu dx^\nu. \tag{1.3.9}$$

We can, therefore, write (1.3.8) simply as:

$$\frac{D}{ds} \frac{d\xi^\varrho}{ds} = 0. \tag{1.3.10}$$

These equations in either form, together with the definition of the line-element ds, fully define the world-line ξ^α as a function of s. These are the equations of a geodesic.

We can also treat W_I as a function of the upper limit of σ_2. This means: we integrate along the geodesic world-line, from a fixed lower limit to an arbitrary upper limit which is the intersection of the world-line with an arbitrary space-like hypersurface. Then W_I becomes identical with the function satisfying the Hamilton-Jacobi equation. In this case we have:

$$\delta W_I = \frac{\partial W_I}{\partial \xi^\alpha} \delta \xi^\alpha = -\Delta m c g_{\alpha\beta} \frac{d\xi^\beta}{ds} \delta \xi^\alpha \bigg|_\sigma, \tag{1.3.11}$$

so that:

$$W_{I|\alpha} = \frac{\partial W_I}{\partial \xi^\alpha} = -\Delta m c g_{\alpha\beta} \frac{d\xi^\beta}{ds}. \tag{1.3.12}$$

Because $g_{\alpha\beta} \dfrac{d\xi^\alpha}{ds} \dfrac{d\xi^\beta}{ds} = 1$, we obtain the Hamilton-Jacobi equation for a test particle:

$$g^{\alpha\beta}(\xi) W_{I|\alpha} W_{I|\beta} = (c\Delta m)^2. \tag{1.3.13}$$

It may sometimes be convenient to find the motion of a particle by finding the total integral of this equation. Differentiating it with respect to the parameters, we may find all the integrals of the equations of motion. In this way we may avoid the sometimes troublesome calculation of the Christoffel symbols.

We shall discuss now the equations of motion of the second type. We treat the field $g_{\alpha\beta}(\xi)$ as unknown; later we shall see how to find it from the field equations. The $g_{\alpha\beta}$ will, in general, be singular on the world-lines. We must therefore be careful with the variation procedure, so as to get rid of the singularities.

We shall vary $\tilde{g}_{\alpha\beta}$ with respect to ξ^{α}. We have:

$$\delta \overset{A}{\overline{g_{\alpha\beta}}} = \int dx \frac{\partial}{\partial \overset{A}{\xi^{\varrho}}} \left(\delta_{(4)}(x - \overset{A}{\xi})\right) g_{\alpha\beta}\, \delta \overset{A}{\xi^{\varrho}}$$

$$= -\int dx\, \delta_{(4)}(x - \overset{A}{\xi})_{|\varrho}\, g_{\alpha\beta}\, \delta \overset{A}{\xi^{\varrho}}$$

$$= \int dx\, \delta_{(4)}(x - \overset{A}{\xi})\, g_{\alpha\beta|\varrho}\, \delta \overset{A}{\xi^{\varrho}} = \overset{A}{\overline{g_{\alpha\beta|\varrho}}}\; \delta \overset{A}{\xi^{\varrho}} \tag{1.3.14}$$

where $g_{\alpha\beta}$ has to be treated as a given function not to be varied with respect to the ξ's.

We introduce

$$ds_A = (\overset{A}{\overline{g_{\alpha\beta}}}\, d\overset{A}{\xi^{\alpha}}\, d\overset{A}{\xi^{\beta}})^{1/2}.$$

(It is convenient to write ds_A instead of $d\overset{A}{s}$.) We then proceed to the variation of W_I:

$$\delta W_I = -\sum_{A=1}^{N} \overset{A}{m}_{(0)} c\, \overset{A}{\overline{g_{\alpha\beta}}} \frac{d\overset{A}{\xi^{\alpha}}}{ds_A}\, \delta \overset{A}{\xi^{\beta}} \Big|_{\sigma_1}^{\sigma_2} +$$

$$+ \sum_{A=1}^{N} \overset{A}{m}_{(0)} c \int_{\sigma_1}^{\sigma_2} ds_A\, \delta \overset{A}{\xi^{\alpha}} \left[\frac{d}{ds_A} \overset{A}{\overline{g_{\alpha\beta}}} \frac{d\overset{A}{\xi^{\beta}}}{ds_A} - \tfrac{1}{2} \overset{A}{\overline{g_{\mu\nu|\alpha}}} \frac{d\overset{A}{\xi^{\mu}}}{ds_A} \frac{d\overset{A}{\xi^{\nu}}}{ds_A} \right]. \tag{1.3.15}$$

From the vanishing of the $\delta\overset{A}{\xi}$ on σ_1 and σ_2 we deduce the equations of motion of the second kind:

$$\frac{d}{ds_A}\overset{A}{\widetilde{g_{\alpha\beta}}}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A} - \tfrac{1}{2}\overset{A}{\widetilde{g_{\mu\nu|\alpha}}}\frac{d\overset{A}{\xi}{}^{\mu}}{ds_A}\frac{d\overset{A}{\xi}{}^{\nu}}{ds_A} = 0. \tag{1.3.16}$$

We may change the form of these equations, remembering that

$$\frac{d}{ds_A}\overset{A}{\widetilde{g_{\alpha\beta}}} = \frac{d}{ds_A}\int dx\, \delta_{(4)}(x-\overset{A}{\xi})\, g_{\alpha\beta}$$

$$= -\frac{d\overset{A}{\xi}{}^{\varrho}}{ds_A}\int dx\, \delta_{(4)}(x-\overset{A}{\xi})_{|\varrho}\, g_{\alpha\beta} \doteq \frac{d\overset{A}{\xi}{}^{\varrho}}{ds_A}\overset{A}{\widetilde{g_{\alpha\beta|\varrho}}}. \tag{1.3.17}$$

Introducing this into (1.3.16) we have:

$$\overset{A}{\widetilde{g_{\alpha\beta}}}\frac{d^2\overset{A}{\xi}{}^{\beta}}{ds_A^2} + \overset{A}{\widetilde{[\mu\nu,\alpha]}}\frac{d\overset{A}{\xi}{}^{\mu}}{ds_A}\frac{d\overset{A}{\xi}{}^{\nu}}{ds_A} = 0. \tag{1.3.18}$$

We shall now make an assumption that is practically always fulfilled, that is:

$$\widetilde{g^{\alpha\beta}g_{\beta\gamma}} = \widetilde{g^{\alpha\beta}}\,\widetilde{g_{\beta\gamma}},$$

$$\widetilde{g^{\alpha\beta}[\mu\nu,\beta]} = \widetilde{g^{\alpha\beta}}\,\widetilde{[\mu\nu,\beta]} = \widetilde{\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}}. \tag{1.3.19}$$

That is, we shall assume the law of the tweedling of products quoted in "Notation" for the g's and their derivatives. Under these conditions we can write (1.3.16) in the form:

$$\overset{A}{\Omega}{}^{\alpha} \equiv \frac{d^2\overset{A}{\xi}{}^{\alpha}}{ds_A^2} + \overset{A}{\widetilde{\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}}}\frac{d\overset{A}{\xi}{}^{\mu}}{ds_A}\frac{d\overset{A}{\xi}{}^{\nu}}{ds_A} = 0. \tag{1.3.20}$$

As in (1.3.9) we can introduce covariant differentiation $\overset{A}{\tilde{D}}$ for tensors on the A'th world-line:

$$\overset{A}{\tilde{D}}T^{\alpha} = dT^{\alpha} + \overbrace{\left\{ {\alpha \atop \mu\nu} \right\}}^{A} T^{\mu} d\overset{A}{\xi}{}^{\nu}. \tag{1.3.21}$$

Using this notation we have

$$\overset{A}{\Omega}{}^{\alpha} = \frac{\overset{A}{\tilde{D}}}{ds_A} \frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A} = 0. \tag{1.3.22}$$

Thus we have obtained the equations in a form very similar to those of a geodesic line. But it must be remembered that the last form was derived on the assumption that $\widetilde{\varphi\psi} = \tilde{\varphi}\tilde{\psi}$ is valid for the g's and their derivatives.

4. THE EINSTEIN FIELD EQUATIONS

We shall recall here the famous Einstein equations for the gravitational field. They can be derived from a variational principle which we postulate in the form $\delta W = 0$ where $W = W_f + W_I$ and

$$W_f = \frac{c^3}{16\pi k} \int\limits_{\sigma_1}^{\sigma_2} dx\sqrt{-g}\,R. \tag{1.4.1}$$

Here k is the gravitational constant, the same that appears in the Newtonian equations:

$$k = 6.67 \cdot 10^{-8}\,\mathrm{cm}^3\,\mathrm{g}^{-1}\,\mathrm{sec}^{-2}.$$

The integral has to be taken between space-like hypersurfaces. The function under the integral is the simplest scalar density that can be formed from the $g_{\alpha\beta}$. Thus it is also the simplest Lagrangian that can be postulated if we ignore the possibility of a constant times $\sqrt{-g}$ which leads to the additional "cosmological term": a constant times $g_{\alpha\beta}$ in the field equations.

Indeed, under certain natural assumptions, $\sqrt{-g}R$ is the only possible Lagrangian. This follows from the fact that the Lagrangian must be a scalar density and that we wish the gravitational field equations to contain derivatives of only up to the second order. This we assume, because the Newtonian equations for the gravitational field are of the second order.

An invariant Lagrangian can be built only from the full Riemannian curvature tensor $R_{\mu\nu\varrho\sigma}$. Therefore, the only scalars to be taken into account besides $R = g^{\mu\nu}R_{\mu\nu}$, are those like $R_{\alpha\beta}R^{\alpha\beta}$, $R_{\mu\nu\varrho\sigma}R^{\mu\nu\varrho\sigma}$, etc. The Lagrange function must be a scalar function of these invariants multiplied by a scalar density. Or, it could be a scalar density in the form $\det\|R_{\alpha\beta}\|$. Generally, all the Lagrangians with the single exception of $\sqrt{-g}R$, lead to field equations of the fourth order. The constant in front of the integral in (1.4.1) is so chosen as to give the transition to Newton's equations for $c \to \infty$.

If there is matter represented by point singularities, then the entire action should be a sum of W_f from (1.4.1) and W_I from (1.3.1):

$$W = W_f + W_I. \tag{1.4.2}$$

The constants $\overset{A}{m}_{(0)}$ in W_I play the role of coupling constants between the point particles and the metric field.

We obtain Einstein's equations for the gravitational field interacting with point particles by varying W with respect to $g_{\alpha\beta}(x)$ and putting $\delta W = 0$.

Let us calculate the variation of W, starting with W_f:

$$\begin{aligned}\delta W_f &= \frac{c^3}{16\pi k}\int_{\sigma_1}^{\sigma_2} dx\,\delta(\sqrt{-g}\,g^{\alpha\beta}R_{\alpha\beta}) \\ &= \frac{c^3}{16\pi k}\int_{\sigma_1}^{\sigma_2} dx(\sqrt{-g}\,R_{\alpha\beta}\,\delta g^{\alpha\beta} + R\,\delta\sqrt{-g} + \sqrt{-g}\,g^{\alpha\beta}\,\delta R_{\alpha\beta}) \\ &= \frac{c^3}{16\pi k}\int_{\sigma_1}^{\sigma_2} dx[\sqrt{-g}(R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}R)\,\delta g^{\alpha\beta} + \sqrt{-g}\,g^{\alpha\beta}\,\delta R_{\alpha\beta}].\end{aligned} \tag{1.4.3}$$

In the above equation we made use of the fact that

$$\delta g = g g^{\alpha\beta} \delta g_{\alpha\beta} = -g g_{\alpha\beta} \delta g^{\alpha\beta}. \tag{1.4.4}$$

Because

$$\delta g^{\alpha\beta} = -g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu}, \tag{1.4.5}$$

we have:

$$\delta W_f = -\frac{c^3}{16\pi k} \int\limits_{\sigma_1}^{\sigma_2} dx \sqrt{-g} (R^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} R) \delta g_{\alpha\beta} + \delta S \tag{1.4.6}$$

where

$$\delta S = \frac{c^3}{16\pi k} \int\limits_{\sigma_1}^{\sigma_2} dx \sqrt{-g}\, g^{\alpha\beta} \delta R_{\alpha\beta}. \tag{1.4.7}$$

To find δS, let us remark that although $\left\{ {\alpha \atop \mu\nu} \right\}$ is not a tensor, its variation $\delta \left\{ {\alpha \atop \mu\nu} \right\}$ is. Indeed $\left\{ {\alpha \atop \mu\nu} \right\} A_\alpha dx^\mu$ is the change of the vector A_ν in a parallel displacement from x^ϱ to $x^\varrho + dx^\varrho$. Therefore $\left(\delta \left\{ {\alpha \atop \mu\nu} \right\}\right) A_\alpha dx^\mu$ is the difference between two vectors at the same point $x^\varrho + dx^\varrho$. Both of them are obtained by a parallel displacement; one of them in the old field $g_{\alpha\beta}$, the other in the varied field $g_{\alpha\beta} + \delta g_{\alpha\beta}$. The difference between two vectors at the same point is a vector. Since A_ν and dx^μ are arbitrary vectors, $\delta \left\{ {\alpha \atop \mu\nu} \right\}$ is a tensor.

Using (0.13) we have:

$$\begin{aligned} g^{\alpha\beta} \delta R_{\alpha\beta} = {} & g^{\alpha\beta} \delta \left(\left\{ {\varrho \atop \beta\varrho} \right\}_{|\alpha} \right) - g^{\alpha\beta} \delta \left(\left\{ {\varrho \atop \beta\alpha} \right\}_{|\varrho} \right) + g^{\alpha\beta} \left\{ {\sigma \atop \beta\varrho} \right\} \delta \left\{ {\varrho \atop \sigma\alpha} \right\} + \\ & + g^{\alpha\beta} \left\{ {\varrho \atop \sigma\alpha} \right\} \delta \left\{ {\sigma \atop \beta\varrho} \right\} - g^{\alpha\beta} \left\{ {\sigma \atop \alpha\beta} \right\} \delta \left\{ {\varrho \atop \sigma\varrho} \right\} - g^{\alpha\beta} \left\{ {\varrho \atop \sigma\varrho} \right\} \delta \left\{ {\sigma \atop \alpha\beta} \right\}. \end{aligned} \tag{1.4.8}$$

Using a locally Galilean coordinate system, that is, one such that at a given point

$$g_{\mu\nu|\varrho} = 0 \quad \text{and therefore} \quad \left\{ {\alpha \atop \mu\nu} \right\} = 0,$$

we have in place of (1.4.8):

$$g^{\alpha\beta} \delta R_{\alpha\beta} = \left(g^{\alpha\mu} \delta \left\{ {\nu \atop \mu\nu} \right\} - g^{\mu\nu} \delta \left\{ {\alpha \atop \mu\nu} \right\} \right)_{|\alpha}. \tag{1.4.9}$$

Because of the tensorial character of the $\delta\left\{\begin{matrix}\alpha\\ \mu\nu\end{matrix}\right\}$ we have in every coordinate system:

$$g^{\alpha\beta}\,\delta R_{\alpha\beta} = \delta Q^{\alpha}{}_{;\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,\delta Q^{\alpha}\right)_{|\alpha} \tag{1.4.10}$$

where

$$\delta Q^{\alpha} = g^{\alpha\mu}\,\delta\left\{\begin{matrix}\nu\\ \mu\nu\end{matrix}\right\} - g^{\mu\nu}\,\delta\left\{\begin{matrix}\alpha\\ \mu\nu\end{matrix}\right\} \tag{1.4.11}$$

is a vector. Therefore, going back to (1.4.7) we find:

$$\delta S = \frac{c^3}{16\pi k}\int_{\sigma_1}^{\sigma_2} dx\left(\sqrt{-g}\,\delta Q^{\alpha}\right)_{|\alpha} = \frac{c^3}{16\pi k}\int_{\sigma} d\mathbf{x}\sqrt{-g}\,\delta Q^{\alpha} n_{\alpha}\Big|_{\sigma_1}^{\sigma_2}. \tag{1.4.12}$$

Assuming that δQ^{α} vanishes on the space-like surfaces σ_1 and σ_2 we have:

$$\delta S = 0. \tag{1.4.12}$$

Therefore, using the definition (0.15) of the Einstein tensor, we can write (1.4.6) in the form:

$$\delta W_f = -\frac{c^3}{16\pi k}\int_{\sigma_1}^{\sigma_2} \mathfrak{G}^{\alpha\beta}\,\delta g_{\alpha\beta}\,dx. \tag{1.4.13}$$

Now let us vary the action W_I with respect to $g_{\alpha\beta}$:

$$\begin{aligned}\delta W_I &= \delta\left[-\sum_{A=1}^{N} \overset{A}{m}_{(0)}c\int_{\sigma_1}^{\sigma_2}\left(\overset{A}{\overline{g_{\alpha\beta}}}\,d\overset{A}{\xi^{\alpha}}\,d\overset{A}{\xi^{\beta}}\right)^{1/2}\right]\\ &= -\sum_{A=1}^{N} \overset{A}{m}_{(0)}c\int_{\sigma_1}^{\sigma_2}\frac{1}{2}\,\frac{d\overset{A}{\xi^{\alpha}}}{ds_A}\,d\overset{A}{\xi^{\beta}}\,\delta\overset{A}{\overline{g_{\alpha\beta}}},\end{aligned} \tag{1.4.14}$$

where

$$\delta\overset{A}{\overline{g_{\alpha\beta}}} = \int dx\,\delta_{(4)}(x-\overset{A}{\xi})\,\delta g_{\alpha\beta}. \tag{1.4.15}$$

Substituting this into (1.4.14) we find:

$$\delta W_I = -\sum_{A=1}^{N} \overset{A}{m}_{(0)} c \int_{\sigma_1}^{\sigma_2} \frac{1}{2} \frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A} d\overset{A}{\xi}{}^{\beta} \int dx\, \delta_{(4)}(x - \overset{A}{\xi})\, \delta g_{\alpha\beta}(x)$$

$$= -\frac{1}{2} \int_{\sigma_1}^{\sigma_2} dx\, \delta g_{\alpha\beta} \sum_{A=1}^{N} \overset{A}{m}_{(0)} c \int_{-\infty}^{\infty} \delta_{(4)}(x - \overset{A}{\xi}) \frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A} \frac{d\overset{A}{\xi}{}^{\beta}}{ds_A} ds_A. \quad (1.4.16)$$

Thus we have performed the variation of W. We demand that δW should vanish, if the $\delta g_{\alpha\beta}$ are arbitrary, subject only to the condition $\delta Q^{\alpha} = 0$ on σ_1 and σ_2 for every choice of the space-like surfaces σ_1 and σ_2. Therefore, putting together the results expressed by (1.4.13) and (1.4.16), we obtain Einstein's equations

$$\mathfrak{G}^{\alpha\beta} = -\frac{8\pi k}{c^2} \sum_{A=1}^{N} \overset{A}{m}_{(0)} \int_{-\infty}^{\infty} \delta_{(4)}(x - \overset{A}{\xi}) \frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A} \frac{d\overset{A}{\xi}{}^{\beta}}{ds_A} ds_A,$$

$$\mathfrak{G}^{\alpha\beta} = \mathfrak{R}^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} \mathfrak{R}. \quad (1.4.17)$$

These equations have the form of field equations representing close-range action. Their left-hand sides consist of non-linear differential operations performed on the metrical field $g_{\alpha\beta}$. Their right-hand sides consist of sources of the field formed by the moving particles. The masses $\overset{A}{m}_{(0)}$ have the character of coupling constants.

Introducing the metrical energy-momentum tensor density for the moving point particles, we have:

$$\mathfrak{T}^{\alpha\beta} = \sum_{A=1}^{N} \overset{A}{m}_{(0)} c^2 \int_{-\infty}^{\infty} \delta_{(4)}(x - \overset{A}{\xi}) \frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A} \frac{d\overset{A}{\xi}{}^{\beta}}{ds_A} ds_A. \quad (1.4.18)$$

With its help we can write the Einstein equations in the simpler and more general form:

$$\mathfrak{G}^{\alpha\beta} = -\frac{8\pi k}{c^4} \mathfrak{T}^{\alpha\beta}. \quad (1.4.19)$$

We assume x^0 to be the parameter for all world-lines:

$$\overset{A}{\xi}{}^k = \overset{A}{\xi}{}^k(x^0), \qquad \overset{A}{\xi}{}^0 = x^0.$$

Then if we introduce

$$\overset{A}{\mu}(x^0) = \frac{k\overset{A}{m}_{(0)}}{c^2}\frac{dx^0}{ds_A} \tag{1.4.22}$$

we can write (1.4.21) in the simple form:

$$\frac{8\pi k}{c^4}\mathfrak{T}^{\alpha\beta} = 8\pi\sum_{A=1}^{N}\overset{A}{\mu}(x^0)\,\delta(\mathbf{x}-\overset{A}{\boldsymbol{\xi}})\,\overset{A}{\xi}{}^{\alpha}{}_{|0}\,\overset{A}{\xi}{}^{\beta}{}_{|0}. \tag{1.4.23}$$

Therefore Einstein's equations (1.4.19) become:

$$\mathfrak{G}^{\alpha\beta} = -8\pi\sum_{A=1}^{N}\overset{A}{\mu}(x^0)\,\delta(\mathbf{x}-\overset{A}{\boldsymbol{\xi}})\,\overset{A}{\xi}{}^{\alpha}{}_{|0}\,\overset{A}{\xi}{}^{\beta}{}_{|0}. \tag{1.4.24}$$

These are the equations which will be especially convenient for finding the equations of motion of the third kind.

One more remark: if we put $x^0 = ct$, then, as follows from (1.4.22), we have:

$$\overset{A}{\mu}(t) = \frac{k\overset{A}{m}_{(0)}}{c^2} + O\left(\frac{1}{c^3}\right). \tag{1.4.25}$$

Therefore, the development of $\overset{A}{\mu}$ in power series of c^{-1} starts with the order two with a constant that has a definite physical meaning, that is with $k\overset{A}{m}_{(0)}c^{-2}$.

5. THE BIANCHI IDENTITIES

For every $g_{\alpha\beta}(x)$ the full Riemannian tensor satisfies the Bianchi identities:

$$R^{\mu}{}_{\nu\varrho\sigma;\tau} + R^{\mu}{}_{\nu\sigma\tau;\varrho} + R^{\mu}{}_{\nu\tau\varrho;\sigma} = 0. \tag{1.5.1}$$

This form is more general since it is supposed to be valid for any energy-momentum tensor.

The left and right-hand sides of these equations are of entirely different character from one another. The left-hand side is Einstein's tensor of the gravitational field. It has a distinct geometrical interpretation, being at the same time a metrical tensor. The right-hand side, however, is a purely physical tensor, representing the distribution of energy and momentum in space-time.

Einstein always regarded this mixture of physics and geometry as fundamentally unsatisfactory, as a temporary device to be replaced in the future by a unified field theory, in which all physical fields would have their geometrical counterparts. To this problem Einstein devoted over thirty years of his life, looking for field equations that would give solutions representing matter free of singularities. The result of this labour performed by him and his many collaborators seems disappointing to most physicists.

For the present, therefore, we assume (1.4.19) and therefore accept a dualistic view; we assume the existence of matter which determines the geometry of the Riemannian continuum. Or, conversely: from the knowledge of the geometry we find the distribution of momentum and energy.

Let us go back to the case of point particles and the definition (1.4.18) of the energy-momentum tensor. This definition means that outside the world-lines we have:

$$\mathfrak{G}^{\alpha\beta} = 0 \qquad (1.4.20)$$

that is, the field equations for empty space. Only on the world-lines does the field become singular. This means: we assume the field equations for empty space and the existence of some world-lines on which the field equations for empty space break down.

For further considerations it will be more convenient to use the energy-momentum tensor for point particles in a slightly different form. We can rewrite (1.4.18) by means of (0.22) in the form:

$$\frac{8\pi k}{c^4}\mathfrak{T}^{\alpha\beta} = \frac{8\pi k}{c^2}\sum_{A=1}^{N} \overset{A}{m}_{(0)} \frac{ds_A}{d\overset{A}{\xi}{}^0}\frac{d\overset{A}{\xi}{}^\alpha}{ds_A}\frac{d\overset{A}{\xi}{}^\beta}{ds_A}\,\delta\big(\mathbf{x}-\overset{A}{\boldsymbol{\xi}}(x^0)\big). \qquad (1.4.21)$$

Since they are tensorial identities it is sufficient to prove their validity in a special coordinate system. Let us take a locally Galilean coordinate system, that is, one for which, at a given point $g_{\alpha\beta} = \eta_{\alpha\beta}$ and $g_{\alpha\beta|\varrho} = 0$, $\left\{ {\alpha \atop \mu\nu} \right\} = 0$, that is a coordinate system for which at a given point x^μ the Christoffel symbols vanish and the g's take the Galilean values. Thus at such a point and in such a coordinate system we can easily find the left-hand side of (1.5.1) from the definition (0.12) of the full Riemannian curvature tensor. Assuming that the derivatives of the third order of the g's are continuous we find that in the chosen coordinate system and at the point x^μ the left-hand side of the last equation vanishes identically. Because of the tensorial character of this equation and because x^μ is an arbitrary point, the left-hand side must be zero always and everywhere.

Multiplying (1.5.1) by $g^{\nu\varrho}$ and putting $\mu = \sigma$, we obtain what are also called the Bianchi identities:

$$G_{\alpha}{}^{\beta}{}_{;\beta} = (R^{\beta}_{\alpha} - \tfrac{1}{2}\delta^{\beta}_{\alpha} R)_{;\beta} \equiv 0. \tag{1.5.2}$$

When referring here or later to Bianchi identities we shall mean equations (1.5.2); they play an important role in G. R. T. and therefore we shall prove them here independently of (1.5.1). They are identities, by which we mean that they are satisfied for arbitrary g's as long as these have continuous third derivatives.

One could ask: what is the source of the Bianchi identities? The answer is: they follow from the covariant character of Einstein's tensor $G^{\alpha\beta}$ and from the existence of a variational principle leading to $G^{\alpha\beta}$.

We shall prove the identities in such a way as to exhibit the reason for their validity, which is: covariance with respect to all transformations.

Once more we write the expression for W_f:

$$W_f = \frac{c^3}{16\pi k} \int_{\sigma_1}^{\sigma_2} dx \sqrt{-g}\, R. \tag{1.5.3}$$

This is obviously a scalar; therefore if we vary the coordinate system, then the corresponding variation of W_f must vanish.

We shall assume an infinitesimally small change in the coordinate system $x^\alpha \to x'^\alpha$, given by:

$$x'^\alpha = x^\alpha - \eta^\alpha \tag{1.5.4}$$

where η^α is an infinitesimally small vector field.

The general transformation law for the g's is:

$$g_{\alpha\beta}(x) = g'_{\mu\nu}(x')\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}. \tag{1.5.5}$$

From this, neglecting everything of order higher than the first, we have:

$$g'_{\alpha\beta}(x) = g_{\alpha\beta}(x) + \delta g_{\alpha\beta}(x) \tag{1.5.6}$$

where

$$\delta g_{\alpha\beta}(x) = g_{\alpha\nu}\eta^\nu{}_{|\beta} + g_{\beta\nu}\eta^\nu{}_{|\alpha} + g_{\alpha\beta|\nu}\eta^\nu \tag{1.5.7}$$

or

$$\delta g_{\alpha\beta} = \eta_{\alpha;\beta} + \eta_{\beta;\alpha}. \tag{1.5.8}$$

Similarly, we obtain

$$\delta g^{\alpha\beta} = -\eta^{\alpha;\beta} - \eta^{\beta;\alpha}. \tag{1.5.9}$$

Remembering (1.4.6) and (1.4.12), we may write δW_f in the form:

$$\delta W_f = -\frac{c^3}{16\pi k}\int_{\sigma_1}^{\sigma_2} dx\,\mathfrak{G}^{\alpha\beta}\delta g_{\alpha\beta} + \frac{c^3}{16\pi k}\int_\sigma d\mathbf{x}\sqrt{-g}\;\delta Q^\alpha n_\alpha\Big|_{\sigma_1}^{\sigma_2} \tag{1.5.10}$$

where δQ^α is given by (1.4.11). Let us now take a field η^α vanishing with its first and second derivatives on the two hypersurfaces σ_1 and σ_2. The surface integrals in (1.5.10) vanish and we obtain

$$\delta W_f = -\frac{c^3}{16\pi k}\int_{\sigma_1}^{\sigma_2} dx\,\mathfrak{G}^{\alpha\beta}\delta g_{\alpha\beta} = 0, \tag{1.5.11}$$

or

$$\delta W_f = -\frac{c^3}{8\pi k}\int_{\sigma_1}^{\sigma_2} dx\, \mathfrak{G}^{\alpha\beta}\eta_{\alpha;\beta} = \frac{c^3}{8\pi k}\int_{\sigma_1}^{\sigma_2} dx\, \mathfrak{G}^{\alpha\beta}{}_{;\beta}\eta_\alpha = 0. \quad (1.5.12)$$

The field η_α being arbitrary in the region between σ_1 and σ_2, we must have

$$\mathfrak{G}^{\alpha\beta}{}_{;\beta} = 0 \quad \text{or} \quad \mathfrak{G}_\alpha{}^\beta{}_{;\beta} = 0. \quad (1.5.13)$$

These equations are, as our proof shows, independent of the field equations. The proof also shows the character of the Bianchi identities. They are a consequence of the invariance of the gravitational action, that is, they are a consequence of the tensorial character of the theory which is covariant with respect to all transformations.

6. THE EQUATIONS OF MOTION ARE A CONSEQUENCE OF THE FIELD EQUATIONS

We return to our fundamental problem. We shall show here that the equations of motion are a consequence of the field equations. This is possible only because the field equations are non-linear (which does not mean, of course, that every non-linear theory gives the equations of motion). Indeed, in the Newtonian theory of gravitation, where the field equations are linear, as well as in Maxwell's theory, the equations of motion are independent of the field equations. In Newtonian theory, we remember, we could find the field for arbitrary motion of the particles which were understood as sources of the field. The equations of motion which we obtained (in the Newtonian case) from the variational principle were independent of the field equations; knowledge of them was unessential for solving the field equations.

The situation is, however, entirely different in G. R. T. The field and motion are intrinsically connected with each other. The

equations of motion are a condition of integrability for the field equations.

We shall now give the mathematical details justifying these remarks and those of Section 1.

Once more we write down Einstein's field equations for a system of point particles:

$$\mathfrak{G}^{\alpha\beta} = -\frac{8\pi k}{c^2}\sum_{A=1}^{N}\overset{A}{m}_{(0)}\int_{-\infty}^{\infty}\delta_{(4)}(x-\overset{A}{\xi})\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}ds_A. \quad (1.4.17)$$

We form the covariant derivative $_{;\beta}$ of the left-hand side. Because of the Bianchi identities we must have as a necessary condition for their integrability

$$0 = \left(\sum_{A=1}^{N}\overset{A}{m}_{(0)}\int_{-\infty}^{\infty}\delta_{(4)}(x-\overset{A}{\xi})\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}ds_A\right)_{;\beta} \quad (1.6.1)$$

Or, briefly, and more generally:

$$\mathfrak{T}^{\alpha\beta}{}_{;\beta} \equiv \mathfrak{T}^{\alpha\beta}{}_{|\beta} + \begin{Bmatrix}\alpha\\ \varrho\beta\end{Bmatrix}\mathfrak{T}^{\varrho\beta} = 0, \quad (1.6.2)$$

where $\mathfrak{T}^{\alpha\beta}$ is the energy-momentum tensor forming the right-hand side of Einstein's field equations. We are here, however, mostly interested in the case of point particles in which $\mathfrak{T}^{\alpha\beta}$ is defined by:

$$\mathfrak{T}^{\alpha\beta} = c^2\sum_{A=1}^{N}\overset{A}{m}_{(0)}\int_{-\infty}^{\infty}\delta_{(4)}(x-\overset{A}{\xi})\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}ds_A. \quad (1.4.18)$$

Thus in this case we have from (1.6.1) and (1.6.2):

$$\frac{1}{c^2}\mathfrak{T}^{\alpha\beta}{}_{;\beta} = \sum_{A=1}^{N}\overset{A}{m}_{(0)}\int_{-\infty}^{\infty}ds_A\left(\delta_{(4)}(x-\overset{A}{\xi})_{|\beta}\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A} + \right.$$

$$\left. + \delta_{(4)}(x-\overset{A}{\xi})\begin{Bmatrix}\alpha\\ \varrho\sigma\end{Bmatrix}\frac{d\overset{A}{\xi}{}^{\varrho}}{ds_A}\frac{d\overset{A}{\xi}{}^{\sigma}}{ds_A}\right) = 0. \quad (1.6.3)$$

This expression may be put into a different form. We see that:

$$\delta_{(4)}(x-\overset{A}{\xi})_{|\beta}\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A} = -\frac{\partial}{\partial\overset{A}{\xi}{}^{\beta}}\delta_{(4)}(x-\overset{A}{\xi})\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}$$

$$= -\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d}{ds_A}\delta_{(4)}(x-\overset{A}{\xi}) = -\frac{d}{ds_A}\left(\frac{\delta\overset{A}{\xi}{}^{\alpha}}{\delta s_A}\delta_{(4)}(x-\overset{A}{\xi})\right)+$$

$$+\delta_{(4)}(x-\overset{A}{\xi})\frac{d^2\overset{A}{\xi}{}^{\alpha}}{ds_A^2}. \qquad (1.6.4)$$

Using this, we may change (1.6.3) into the form

$$\int f(x)\,\delta_{(4)}(x-\overset{A}{\xi})\,ds$$

since the integral of $\dfrac{d}{ds_A}\left(\dfrac{d\xi^{\alpha}}{ds_A}\delta_{(4)}\right)$ is equal to zero for every finite x. Furthermore, expressions of the type $f\delta_{(4)}$ vanish everywhere with the exception of the point at which $\delta_{(4)}$ does not vanish. Thus we may write:

$$f(x)\,\delta_{(4)}(x-\overset{A}{\xi}) = \overset{A}{\tilde{f}}\,\delta_{(4)}(x-\overset{A}{\xi}).$$

We can therefore write (1.6.3) in the form:

$$\frac{1}{c^2}\mathfrak{T}^{\alpha\beta}{}_{;\beta} = \sum_{A=1}^{N}\overset{A}{m}_{(0)}\int_{-\infty}^{\infty}ds_A\,\delta_{(4)}(x-\overset{A}{\xi})\left[\frac{d^2\overset{A}{\xi}{}^{\alpha}}{ds_A^2}+\overline{\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}}^{A}\frac{d\overset{A}{\xi}{}^{\mu}}{ds_A}\frac{d\overset{A}{\xi}{}^{\nu}}{ds_A}\right] = 0. \quad (1.6.5)$$

Or in the form:

$$\frac{1}{c^2}\mathfrak{T}^{\alpha\beta}{}_{;\beta} = \sum_{A=1}^{N}\overset{A}{m}_{(0)}\int_{-\infty}^{\infty}ds_A\,\delta_{(4)}(x-\overset{A}{\xi})\overset{A}{\Omega}{}^{\alpha}(s_A) = 0 \qquad (1.6.6)$$

where, as in (1.3.20):

$$\overset{A}{\Omega}{}^{\alpha} \equiv \frac{d^2\overset{A}{\xi}{}^{\alpha}}{ds_A^2}+\overline{\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}}^{A}\frac{d\overset{A}{\xi}{}^{\mu}}{ds_A}\frac{d\overset{A}{\xi}{}^{\nu}}{ds_A}. \qquad (1.3.20)$$

We see that the integrability conditions (1.6.6) are fulfilled if

$$\overset{A}{\Omega}{}^{\alpha} = 0. \tag{1.6.7}$$

But the converse theorem is also true. This means: from the vanishing of the divergence of the energy-momentum tensor, one can deduce the equations of motion. Indeed, let $\Delta\sigma_B$ be a finite three-dimensional region of space-like character (that is a piece of the space-like hypersurface σ_B). We assume further that this piece is cut by the line $\overset{B}{\xi}{}^{\alpha} = \overset{B}{\xi}{}^{\alpha}(s_B)$ and only by this line; the point of intersection is, say, $s_B = \bar{s}_B$. If $\mathfrak{T}^{\alpha\beta}{}_{;\beta} = 0$, then we also have

$$\frac{d\overset{B}{\xi}{}^{\varrho}}{ds_B}\bigg|_{s_B=\bar{s}_B} \int\limits_{\Delta\sigma_B} d\mathbf{x}\, n_{\varrho} \mathfrak{T}^{\alpha\beta}{}_{;\beta} = 0. \tag{1.6.8}$$

Because of (1.6.6) this means:

$$\frac{d\overset{B}{\xi}{}^{\varrho}}{ds_B}\bigg|_{s_B=\bar{s}_B} \int\limits_{\Delta\sigma_B} d\mathbf{x}\, n_{\varrho} \overset{B}{m}_{(0)} \int\limits_{-\infty}^{\infty} ds_B\, \delta_{(4)}(x - \overset{B}{\xi})\overset{B}{\Omega}{}^{\alpha}(s_B) = 0. \tag{1.6.9}$$

Taking into account the properties of the $\delta_{(4)}$-function, the above equation means:

$$\overset{B}{m}_{(0)}\overset{B}{\Omega}{}^{\alpha}(\bar{s}_B) = 0. \tag{1.6.10}$$

As this equation must be fulfilled for every $\bar{s}_B$, we see that indeed the vanishing of the energy-momentum tensor implies the validity of the equations of motion.

In the proof, we used the covariant notation. By giving up the covariant notation, we simplify the proof. We can write (1.6.6) using δ instead of $\delta_{(4)}$:

$$\frac{1}{c^2}\mathfrak{T}^{\alpha\beta}{}_{;\beta} = \sum_{A=1}^{N} \overset{A}{m}_{(0)}\, \delta(\mathbf{x} - \overset{A}{\boldsymbol{\xi}})\overset{A}{\Omega}{}^{\alpha} \frac{ds_A}{d\overset{A}{\xi}{}^{0}} = 0. \tag{1.6.11}$$

Integrating over a three-dimensional region V around the point $\overset{B}{\xi}$, we see immediately that

$$\frac{1}{c^2}\frac{d\overset{B}{\xi}{}^0}{ds_B}\int\limits_V d\mathbf{x}\,\mathfrak{T}^{\alpha\beta}{}_{;\beta} = \overset{B}{m}_{(0)}\overset{B}{\Omega}{}^\alpha(x^0) = 0. \tag{1.6.12}$$

Thus, again we obtain the equations of motion as consistency conditions for the field equations.

We have obtained the equations of motion in two ways: firstly, by varying W_f+W_I (or really only W_I) with respect to the world-lines; secondly, by finding the integrability conditions for the field equations. What has been said up to now may be summarized in the following table in which $\delta\ldots/\delta\ldots$ denotes the variational derivatives:

$$\frac{\delta}{\delta\overset{A}{\xi}}(W_f+W_I) = \frac{\delta W_I}{\delta\overset{A}{\xi}} = 0 \rightarrow \text{Equations of motion}$$

$$\uparrow$$

$$\frac{\delta}{\delta g^{\alpha\beta}}(W_f+W_I) = 0 \rightarrow \text{Field equations} \tag{1.6.13}$$

Thus we see the essential difference between Newton's and Einstein's theories of gravitation. To find the world-lines in G. R. T. we have to know the field. But to find the field we have to know the motion. It cannot be arbitrary.

What is the way out of this difficulty? We shall see that there are two ways out. The one mentioned in Section 1 is useful in principle rather than in practice. It consists of changing Einstein's equations so that the motion is arbitrary. Then, imposing proper restrictions upon the motion, we obtain the solution of the equations. The other, which we mentioned in Section 1, consists of devising an adequate approximation procedure. We shall have much more to say about each of these methods later.

7. FORMS FOR THE EQUATIONS OF MOTION OF THE SECOND KIND

As we have seen in the previous section, equations of motion of the second kind are closely connected with the Bianchi identities. These identities follow from the invariance of gravitational action with respect to infinitesimal variation; this we have proved with the help of formula (1.5.13).

Now, making use of the field equations in (1.5.11), we obtain:

$$\int dx\,\mathfrak{T}^{\alpha\beta}\delta g_{\alpha\beta} = 0 \qquad \text{or} \qquad \int dx\,\mathfrak{T}_{\alpha\beta}\delta g^{\alpha\beta} = 0, \tag{1.7.1}$$

where according to (1.5.8) and (1.5.9):

$$-\delta g^{\alpha\beta} = g^{\alpha\mu}\eta^{\beta}{}_{|\mu} + g^{\beta\mu}\eta^{\alpha}{}_{|\mu} - g^{\alpha\beta}{}_{|\mu}\eta^{\mu}, \tag{1.7.2a}$$

$$\delta g_{\alpha\beta} = g_{\alpha\mu}\eta^{\mu}{}_{|\beta} + g_{\beta\mu}\eta^{\mu}{}_{|\alpha} + g_{\alpha\beta|\mu}\eta^{\mu}. \tag{1.7.2b}$$

If we introduce the last values into (1.7.1) we see that because η is arbitrary, (1.7.1) is equivalent to

$$\mathfrak{T}^{\alpha\beta}{}_{;\beta} = 0 \qquad \text{or} \qquad \mathfrak{T}_{\alpha;\beta}^{\beta} = 0. \tag{1.7.3}$$

Thus, instead of (1.7.3) we can equally well regard equations (1.7.1) as our equations of motion, just because of the arbitrary character of the η's.

Let us introduce into the first of the equations (1.7.1) for $\mathfrak{T}^{\alpha\beta}$ the known expression (1.4.23):

$$\frac{k}{c^4}\mathfrak{T}^{\alpha\beta} = \sum_{A=1}^{N} \overset{A}{\mu}\,\overset{A}{\delta}\,\overset{A}{\xi}{}^{\alpha}{}_{|0}\,\overset{A}{\xi}{}^{\beta}{}_{|0}. \tag{1.7.4}$$

Therefore (1.7.1) becomes:

$$\sum_{A=1}^{N} \int_{x'^0}^{x''^0} dx^0\,\overset{A}{\mu}\,\overset{A}{\xi}{}^{\alpha}{}_{|0}\,\overset{A}{\xi}{}^{\beta}{}_{|0}\,\overline{\delta g_{\alpha\beta}} = 0. \tag{1.7.5}$$

Since the η's are continuous functions of the x's only (not depending on the ξ's), we have:

$$\overset{A}{\overline{\delta g_{\alpha\beta}}} = \overset{A}{\overline{g_{\alpha\mu}}}\,\overset{A}{\overline{\eta^{\mu}{}_{|\beta}}} + \overset{A}{\overline{g_{\beta\mu}}}\,\overset{A}{\overline{\eta^{\mu}{}_{|\alpha}}} + \overset{A}{\overline{g_{\alpha\beta|\mu}}}\,\overset{A}{\overline{\eta^{\mu}}}. \tag{1.7.6}$$

Since

$$\overset{A}{\tilde{\eta}}{}^{\varrho}{}_{|\alpha}\overset{A}{\xi}{}^{\alpha}{}_{|0} = \frac{d\overset{A}{\tilde{\eta}}{}^{\varrho}}{dx^{0}}, \tag{1.7.7}$$

we have, because of the arbitrary character of the $\overset{A}{\tilde{\eta}}$'s which, as we assume, vanish at the end of the time interval:

$$\frac{d}{dx^{0}}(\overset{A}{\mu}\overset{A}{\tilde{g}}_{\alpha\beta}\overset{A}{\xi}{}^{\beta}{}_{|0}) - \tfrac{1}{2}\overset{A}{\mu}\overset{A}{\tilde{g}}_{\mu\nu|\alpha}\overset{A}{\xi}{}^{\mu}{}_{|0}\overset{A}{\xi}{}^{\nu}{}_{|0} = 0, \qquad A = 1, \ldots, N. \tag{1.7.8}$$

Let us first show that these equations are entirely equivalent to those gained by the use of the variational principle, that is to (1.3.16)

$$\frac{d}{ds_A}\left(\overset{A}{\tilde{g}}_{\alpha\beta}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}\right) - \frac{1}{2}\overset{A}{\tilde{g}}_{\mu\nu|\alpha}\frac{d\overset{A}{\xi}{}^{\mu}}{ds_A}\frac{d\overset{A}{\xi}{}^{\nu}}{ds_A} = 0. \tag{1.7.9}$$

Before we prove this equivalence, let us make one remark: these equations are typical for the two formalisms which are used throughout our work: the s formalism, if s is an independent parameter, as in the last equation, and the t formalism, if t is the independent variable, as in (1.7.8). The s formalism connected with the use of $\delta_{(4)}(x)$ has the virtue of revealing immediately the covariant character of these equations; the t formalism, connected with the use of $\delta(\mathbf{x})$, does not have this virtue. It is, however, easier to handle in actual calculations.

The proof of the equivalence of the last two equations is simple enough. Indeed, introducing into (1.7.9) instead of ds_A:

$$ds_A = \frac{k\overset{A}{m}_{(0)}}{c^2}\frac{dx^0}{\overset{A}{\mu}(x^0)}, \tag{1.7.10}$$

(which is identical with (1.4.22)), we easily find that (1.7.9) becomes (1.7.8).

The reverse statement is also true, but the proof of it is more subtle. It would seem that by introducing into (1.7.8) the new parameter s_A defined through the last equation we do obtain (1.7.9). This is true, but the remark does not make the proof complete. The question arises: is the ds_A introduced through the last equation identical with the ds_A introduced through

$$\bar{g}_{\alpha\beta}\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A} = 1? \tag{1.7.11}$$

We shall prove that it is. In fact, differentiating the last equation with respect to s_A, we obtain according to (0.18)

$$\frac{d}{ds_A}\left(\overset{A}{\bar{g}}_{\alpha\beta}\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}\right) = \overset{A}{\bar{g}}_{\alpha\beta|\varrho}\frac{d\overset{A}{\xi}{}^{\varrho}}{ds_A}\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A} + $$

$$+2\overset{A}{\bar{g}}_{\alpha\beta}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}\frac{d^2\overset{A}{\xi}{}^{\alpha}}{ds_A{}^2} = 0. \tag{1.7.12}$$

Obviously, as we mentioned before, the substitution of the new parameter ds_A defined by (1.7.10) turns the form of equation (1.7.8) into the form (1.7.9). But in it, ds_A is the "proper time" since it satisfies (1.7.12). This can be seen in the usual way by multiplying (1.7.9) by $d\overset{A}{\xi}{}^{\alpha}/ds_A$. In this way we obtain an equation entirely equivalent to the last one, from which it follows that

$$\overset{A}{\bar{g}}_{\alpha\beta}\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A} = \overset{A}{\text{const}} \tag{1.7.13}$$

is an integral of the differential equations for this world-line.

In the definition (1.7.10) of ds_A we have used the constant $\overset{A}{m}_{(0)}$. Obviously it can always be so chosen that the constant in the last equation equals one. This completes our proof.

Both these equations of motion can obviously be deduced from a Lagrangian: the first one, (1.7.8), from a variational principle

$$\left.\begin{aligned} &\delta \int L\,dx^0 = 0, \\ &L = \sum_A \int g_{\alpha\beta} \overset{A}{\mu} \overset{A}{\xi}{}^{\alpha}{}_{|0} \overset{A}{\xi}{}^{\beta}{}_{|0}\, \delta(\mathbf{x} - \overset{A}{\xi})\, d\mathbf{x}; \end{aligned}\right\} \tag{1.7.14}$$

the second one, (1.7.9), from

$$\left.\begin{aligned} &\delta\Big(-\sum_A \overset{A}{m}_{(0)}\, c \int ds_A\Big) = 0, \\ &ds_A^2 = \int g_{\alpha\beta}\, d\overset{A}{\xi}{}^{\alpha} d\overset{A}{\xi}{}^{\beta}\, \delta_{(4)}\, dx. \end{aligned}\right\} \tag{1.7.15}$$

In both cases, the g's should not be varied with respect to the ξ's.

In exactly the same way in which we passed from (1.3.16) to (1.3.20), that is by raising the indices, we can obtain from our two equations, again assuming the rule for tweedling the products:

$$(\overset{A}{\mu}\overset{A}{\xi}{}^{\alpha}{}_{|0})_{|0} + \overset{A}{\mu} \overset{A}{\overline{\left\{ {\alpha \atop \nu\mu} \right\}}} \overset{A}{\xi}{}^{\mu}{}_{|0} \overset{A}{\xi}{}^{\nu}{}_{|0} = 0, \tag{1.7.16}$$

and, as in (1.3.20), we obtain:

$$\frac{d^2 \overset{A}{\xi}{}^{\alpha}}{ds_A^2} + \overset{A}{\overline{\left\{ {\alpha \atop \mu\nu} \right\}}} \frac{d\overset{A}{\xi}{}^{\mu}}{ds_A} \frac{d\overset{A}{\xi}{}^{\nu}}{ds_A} = 0. \tag{1.7.17}$$

Note that equation (1.7.8) could also be obtained from

$$\mathfrak{T}_{\alpha}{}^{\beta}{}_{;\beta} = 0 \tag{1.7.18}$$

by putting into this equation the expression corresponding to that in (1.7.4) and following an argument similar to the one employed in the previous section where the equation

$$\mathfrak{T}^{\alpha\beta}{}_{;\beta} = 0 \tag{1.7.19}$$

was employed. But obviously we wish to get the same equations of motion whether we use the lower or the mixed indices in the

field equations, from which we deduce either (1.7.18) or (1.7.19) respectively. These two forms of equations are equivalent only if

$$\overbrace{g^{\alpha\varrho}g_{\varrho\beta}}^{A} = \overbrace{g^{\alpha\varrho}}^{A}\,\overbrace{g_{\varrho\beta}}^{A} \tag{1.7.20}$$

$$\overbrace{g^{\alpha\varrho}[\mu\nu,\varrho]}^{A} = \overbrace{g^{\alpha\varrho}}^{A}\,\overbrace{[\mu\nu,\varrho]}^{A}. \tag{1.7.21}$$

Thus only if the tweedling laws for products are assumed, are the field equations ot G. R. T. self-consistent.

We shall introduce one more form of the equations of motion which is of some practical value, namely: in equations (1.7.16) there are $4N$ unknown $\overset{A}{\xi}{}^{k}$ and $\overset{A}{\mu}$. From them we can eliminate the μ's and obtain $3N$ equations for the $3N$ unknown $\overset{A}{\xi}{}^{k}$.

Putting $\alpha = 0$ and $\alpha = a$ in (1.7.16), we have:

$$\overset{A}{\mu}{}_{|0} + \overset{A}{\mu}\,\overbrace{\begin{Bmatrix}0\\ \mu\nu\end{Bmatrix}}^{A}\,\overset{A}{\xi}{}^{\mu}{}_{|0}\,\overset{A}{\xi}{}^{\nu}{}_{|0} = 0, \tag{1.7.22}$$

$$\overset{A}{\mu}{}_{|0}\,\overset{A}{\xi}{}^{a}{}_{|0} + \overset{A}{\mu}\,\overset{A}{\xi}{}^{a}{}_{|00} + \overset{A}{\mu}\,\overbrace{\begin{Bmatrix}a\\ \mu\nu\end{Bmatrix}}^{A}\,\overset{A}{\xi}{}^{\mu}{}_{|0}\,\overset{A}{\xi}{}^{\nu}{}_{|0} = 0. \tag{1.7.23}$$

Eliminating $\overset{A}{\mu}{}_{|0}$ from the second equation with the help of the first, we have the three equations for the A'th particle:

$$\overset{A}{\xi}{}^{a}{}_{|00} + \left(\overbrace{\begin{Bmatrix}a\\ \mu\nu\end{Bmatrix}}^{A}\,\overset{A}{\xi}{}^{0}{}_{|0} - \overbrace{\begin{Bmatrix}0\\ \mu\nu\end{Bmatrix}}^{A}\,\overset{A}{\xi}{}^{a}{}_{|0}\right)\overset{A}{\xi}{}^{\mu}{}_{|0}\,\overset{A}{\xi}{}^{\nu}{}_{|0} = 0. \tag{1.7.24}$$

This we can write more explicitly though less symmetrically:

$$\begin{aligned}\overset{A}{\xi}{}^{a}{}_{|00} + \overbrace{\begin{Bmatrix}a\\ 00\end{Bmatrix}}^{A} - \overbrace{\begin{Bmatrix}0\\ 00\end{Bmatrix}}^{A}\,\overset{A}{\xi}{}^{a}{}_{|0} + 2\overbrace{\begin{Bmatrix}a\\ 0n\end{Bmatrix}}^{A}\,\overset{A}{\xi}{}^{n}{}_{|0} - 2\overbrace{\begin{Bmatrix}0\\ 0n\end{Bmatrix}}^{A}\,\overset{A}{\xi}{}^{a}{}_{|0}\,\overset{A}{\xi}{}^{n}{}_{|0} + {}&\\ + \overbrace{\begin{Bmatrix}a\\ mn\end{Bmatrix}}^{A}\,\overset{A}{\xi}{}^{m}{}_{|0}\,\overset{A}{\xi}{}^{n}{}_{|0} - \overbrace{\begin{Bmatrix}0\\ mn\end{Bmatrix}}^{A}\,\overset{A}{\xi}{}^{a}{}_{|0}\,\overset{A}{\xi}{}^{m}{}_{|0}\,\overset{A}{\xi}{}^{n}{}_{|0} = 0.&\end{aligned} \tag{1.7.25}$$

Thus the mass $\overset{A}{\mu}(x^0)$ can be made to disappear from the equations in which x^0 is used as a parameter. This fact suggests some remarks of a more general nature.

In Newtonian Theory, the equations of motion of the first and second kind had one characteristic feature: it was possible to cross out from both sides of the equations the mass of the particle whose motion was considered. The mass on the left-hand side, appearing as the coefficient of acceleration, was the inertial mass, whereas the mass on the right-hand side, which was always multiplied by the mass of another particle, was the gravitational mass. This fact was the mathematical expression of a truth known since the times of Galileo: that all bodies fall in empty space with the same acceleration. This was confirmed again by the more precise experiments of Roland Eötvös which showed that the inertial mass and the gravitational mass are numerically equal.

In our presentation of Newtonian Theory, this physical truth found expression in Section 6. In writing the Newtonian action W_I we used the same coefficients $\overset{A}{m}$ for the inertial constants as for the coupling constants with the gravitational field. That is: in $\sum_A \frac{1}{2}\overset{A}{m}\,\dot{\overset{A}{\xi}}{}^a \dot{\overset{A}{\xi}}{}^a$ we used the inertial constants and in $-\sum_A \overset{A}{m}\int d\mathbf{x}\,\delta(\mathbf{x}-\overset{A}{\boldsymbol{\xi}})\,\varphi$, the gravitational constants. There was no *a priori* reason to regard the m's as identical in these expressions. We were forced to do so only because of the experiments of Galileo and Eötvös. But we can well imagine a world in which the inertial and gravitational masses are different from one another.

In G. R. T., however, the equivalence of gravitational and inertial mass is embodied naturally into the foundations of the theory. Indeed, unlike Newtonian Theory, we have in G. R. T.:

$$W_I = -\sum_{A=1}^{N} \overset{A}{m}_{(0)}\, c \int_{\sigma_1}^{\sigma_2} (\overset{A}{\overline{g}}_{\alpha\beta}\, d\overset{A}{\xi}{}^{\alpha}\, d\overset{A}{\xi}{}^{\beta})^{1/}$$

and there is no place here for the distinction between the inertial part and that representing the interaction with the field. Thus, there is no place for a distinction between inertial and gravitational mass. As we shall see in Chapter II, in the first approximation, with respect to $1/c$, the expression W_I in G. R. T. consists of two parts: one of them representing the inertial part, the second the interaction. But in both parts we do have identical masses, not because of a special assumption, but as a consequence of G. R. T.

One of the results of the equality between inertial and gravitational mass is that in the equations of motion of the second kind the mass $\overset{A}{m}_{(0)}$ does not appear at all. What does this mean physically? It means that the acceleration at a given point depends only on the field at the point where this particle is situated, and not on its mass. The fact that the equations of motion of the second kind are externally independent of $\overset{A}{m}_{(0)}$ is the mathematical expression of the equivalence principle in G. R. T.

The same is of course true for equations of motion of the first kind, that is for the equations of motion for a test particle. Then, of course, the law of the tweedling of products is fulfilled and the entire tweedling procedure becomes a trivial matter of putting the ξ's in place of the x's. Thus our equations of motion of the second kind, changing into equations of motion of the first kind, rigorously become those of a geodesic line. They are entirely independent of the mass of the particle, not only externally; the field itself is free from any dependence on $\Delta\overset{A}{m}$, as long as it is small enough not to disturb the external field.

In the natural explanation of the equality between gravitational and inertial mass, we see the great and essential superiority of G. R. T. over other gravitational theories.

The difference between the gravitational and inertial mass appears only if we consider equations of the third kind. More about this in the last chapter.

8. THE EQUATIONS OF MOTION IN GRAVITATIONAL AND NON-GRAVITATIONAL FIELDS

Up to now we have considered gravitational equations with the energy-momentum tensor

$$\mathfrak{T}^{\alpha\beta} = \sum_{A=1}^{N} \overset{A}{m}_{(0)} c^2 \int_{-\infty}^{\infty} \delta_{(4)}(x - \overset{A}{\xi}) \frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A} \frac{d\overset{A}{\xi}{}^{\beta}}{ds_A} ds_A \tag{1.8.1}$$

We shall now briefly consider the more general case, in which some fields are added, say the electromagnetic or the mesonic field, or both. Let us assume, only for the sake of brevity, that one vectorial field is added, denoted by A^{α}. (Of course, everything that is said here remains true for tensor or spinor fields.) Then the gravitational equations change their form. Besides the energy-momentum tensor belonging to the particles, there also appears a tensor belonging to the field. If we call this additional tensor $S^{\alpha\beta}$ and the entire tensor $E^{\alpha\beta}$, then Einstein's equations take the form:

$$\mathfrak{G}^{\alpha\beta} = -\frac{8\pi k}{c^4}(\mathfrak{T}^{\alpha\beta} + \mathfrak{S}^{\alpha\beta}) = -\frac{8\pi k}{c^4}\mathfrak{E}^{\alpha\beta}. \tag{1.8.2}$$

We do not need to care about the actual form of $S^{\alpha\beta}$. It is sufficient for us to know that $S^{\alpha\beta}$ is a tensor formed of the A^{α}, perhaps of their covariant derivatives and it may also depend explicitly on the metric tensor and on the velocity vectors. Thus, symbolically, we may write:

$$S^{\alpha\beta} = S^{\alpha\beta}(A, g, \xi'). \tag{1.8.3}$$

The metric tensor and the equations of motion are determined by (1.8.2), if the external field A^{α} is known. As to the A's, they must be determined by further field equations. Since we do not care here about their specific form, we may write them symbolically in the following way:

$$O_{\alpha}(A, g, \xi') = 0, \tag{1.8.4}$$

where the O_{α} are some operations to be performed on the A, g fields and on the velocity vectors. Our only assumption concerning

the operators O_α is that equations (1.8.4) should be covariant with respect to arbitrary transformations. Usually both equations (1.8.2) for the gravitational field and (1.8.4) for the non-gravitational field can be deduced from a Lagrangian, the first by a variation of the metric tensor, and the second by a variation of the external field A.

Obviously, because of the Bianchi identities we have:

$$\mathfrak{E}^{\alpha\beta}{}_{;\beta} = \mathfrak{T}^{\alpha\beta}{}_{;\beta} + \mathfrak{S}^{\alpha\beta}{}_{;\beta} = 0. \tag{1.8.5}$$

Now there are two possibilities:

$$1.\ \mathfrak{S}^{\alpha\beta}{}_{;\beta} = 0, \qquad 2.\ \mathfrak{S}^{\alpha\beta}{}_{;\beta} \neq 0.$$

The first possibility may be a result of the field equations for the A's. To have a definite example before our eyes let us think about the electromagnetic tensor without sources. Then $\mathfrak{S}^{\alpha\beta}{}_{;\beta} = 0$ because of Maxwell's equations. In this case we have, for the equations of motion:

$$\frac{1}{c^2}\frac{d\xi^0}{ds_A}\int\limits_{\overset{A}{\Omega}} \mathfrak{T}^{\alpha\beta}{}_{;\beta}\,d\mathbf{x} = 0, \qquad A = 1, 2, \ldots, N, \tag{1.8.6}$$

exactly as before. In this case the equations of motion are influenced only by the change of the metric field, a change caused by the existence of the A field. But in their structure the equations remain unchanged; they are the equations of the "geodesic line", where the $g_{\alpha\beta}$, $g_{\alpha\beta,\varrho}$ are replaced by $\widetilde{g_{\alpha\beta}}$, $\widetilde{g_{\alpha\beta,\varrho}}$.

The second possibility gives us different equations of motion. They are now:

$$\mathfrak{E}^{\alpha\beta}{}_{;\beta} = \mathfrak{T}^{\alpha\beta}{}_{;\beta} + \mathfrak{S}^{\alpha\beta}{}_{;\beta} = 0. \tag{1.8.7}$$

An interesting case arises when the vector density $S^{\alpha\beta}{}_{;\beta}$ is a linear function of the δ's:

$$-\mathfrak{S}^{\alpha\beta}{}_{;\beta} = c^2 \sum_{A=1}^{N} \overset{A}{K}{}^{\alpha} \frac{ds_A}{d\xi^0} \overset{A}{\delta}_{(3)}. \tag{1.8.8}$$

This is so, for example, in the case of electromagnetic fields in which the particles of mass $\overset{A}{m}_{(0)}$ have a charge $\overset{A}{e}$. In such a case, characterized generally by (1.8.8), we have for the equations of motion:

$$\overset{A}{m}_{(0)} \frac{d^2 \overset{A}{\xi}{}^{\alpha}}{ds_A^2} + \overset{A}{m}_{(0)} \overline{\left\{ {\alpha \atop \mu\nu} \right\}}^{A} \frac{d\overset{A}{\xi}{}^{\mu}}{ds_A} \frac{d\overset{A}{\xi}{}^{\nu}}{ds_A} = \overset{A}{K}{}^{\alpha}, \qquad A = 1, \ldots, N. \quad (1.8.9)$$

These are the general equations of motion if (1.8.8) is assumed. In particular, in the case of an electromagnetic field, we have the Lorentz equations of motion.

Usually, (1.8.8) will be valid in the case of point particles, as a consequence of the field equations (1.8.4). Thus in such a case we may say that the proper equations of motion are a consequence of the field equations (for the A's) and the gravitational equations. But of course the same equations of motion are valid in an arbitrary coordinate system, even if the gravitational field vanishes, that is if the gravitational constant $k \to 0$ and $R^{\mu}{}_{\nu\varrho\sigma} = 0$. In this case the equations of motion (1.8.9) are usually assumed, because the field equations for the A's usually become linear and, as we know, a linear theory cannot give the proper equations of motion. They have to be added to the field equations. We see the advantage of introducing the gravitational field. Even when it goes to zero, a trace remains in the form of the equations of motion. Otherwise such equations would have to be assumed separately.

It can easily be shown that the K^{α} vector in (1.8.8) must be a space-like vector. This can be seen by multiplying the last equation by

$$\overline{g}^{A}_{\alpha\beta} \frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}. \quad (1.8.10)$$

The left-hand side of the resulting equation must vanish. This is so because of the definition of ds_A and because of the considerations in Section 7. Therefore the right-hand side must vanish too.

Thus we have:

$$\overset{A}{g}_{\alpha\beta}\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\overset{A}{K}{}^{\beta} = 0 \tag{1.8.11}$$

which proves our point, since the velocity is a time-like vector.

9. EQUATIONS OF MOTION IN DIFFERENT COORDINATE SYSTEMS

We shall answer the following question: what happens to the field equations and to the equations of motion if we pass from one coördinate system to another? As in the last section, we assume the existence of the gravitational field and of the physical field characterized, as before, for the sake of brevity, by one vector field A^{α}. Now we pass from a given coordinate system to a starred coordinate system, that is from

$$x^{\alpha},\ g^{\alpha\beta}(x),\ A^{\alpha}(x),\ \xi^{\alpha} \tag{1.9.1}$$

to the expressions

$$x^{*\alpha},\ g^{*\alpha\beta}(x^{*}),A^{*\alpha}(x^{*}),\ \xi^{*\alpha}, \tag{1.9.2}$$

where

$$x^{\alpha} = x^{\alpha}(x^{*}),\qquad x^{*\alpha} = x^{*\alpha}(x),$$

$$A^{*\alpha} = A^{\sigma}\frac{\partial x^{*\alpha}}{\partial x^{\sigma}},\ g^{*\alpha\beta} = g^{\varrho\sigma}\frac{\partial x^{*\alpha}}{\partial x^{\varrho}}\frac{\partial x^{*\beta}}{\partial x^{\sigma}},\ \text{etc.} \tag{1.9.3}$$

Similarly the expressions

$$A^{\alpha}{}_{;\beta} = A^{\alpha}{}_{|\beta} + \begin{Bmatrix}\alpha\\ \beta\sigma\end{Bmatrix}A^{\sigma} \tag{1.9.4}$$

go over into

$$A^{*\alpha}{}_{\overset{*}{;\beta}} = A^{*\alpha}{}_{|\beta} + \begin{Bmatrix}\alpha\\ \beta\sigma\end{Bmatrix}^{*}A^{*\sigma} \tag{1.9.5}$$

where the starred Christoffel symbols have the obvious meaning.

But the field equations of General Relativity Theory cannot depend on the choice of the coordinate system. This means that, whereas in the old coordinate system we had field equations that could be written symbolically in the form:

$$\left.\begin{aligned} G^{\alpha\beta}[g(x)] &= -\frac{8\pi k}{c^4} E^{\alpha\beta}[A(x), g(x), \xi'], \\ O_\alpha(x)[A(x), g(x), \xi'] &= 0, \end{aligned}\right\} \tag{1.9.6}$$

in the new coordinate system the equations can be written in the form:

$$\left.\begin{aligned} G^{\alpha\beta}[g^*(x^*)] &= -\frac{8\pi k}{c^4} E^{\alpha\beta}[A^*(x^*), g^*(x^*), \xi^{*\prime}], \\ O_\alpha(x^*)[A^*(x^*), g^*(x^*), \xi^{*\prime}] &= 0. \end{aligned}\right\} \tag{1.9.7}$$

All this is almost trivial, yet the fact that in the new coordinate system the $G^{\alpha\beta}$, the $E^{\alpha\beta}$ and the operator O_α depend on the new field components and act in the same way as they depended and acted before in the old coordinate system on the old field components, is not always well enough understood.

From this it follows that the equations of motion in the new coordinate system are

$$E^{\alpha\beta}(A^*(x^*), g^*(x^*), \xi^{*\prime})_{\overset{*}{;}\beta} = 0 \tag{1.9.8}$$

where the star over the semicolon means, as in (1.9.5), that the Christoffel symbols have to be formed from the $g^*(x^*)$.

Again the result is almost trivial: the equations of motion are covariant with respect to any change in the coordinate system.

Since the field equations and those of motion have the same form in all coordinate systems, then the question arises: are all these coordinate systems permissible? But to describe motion in any particular instance we must specify the coordinate system in some way. Yet by doing so we violate the spirit of covariance, so important in G. R. T.

In principle there are three ways out of this difficulty:

1. The coordinate system is specified by certain added coordinate conditions. Indeed, in general four coordinate conditions can be added to the field equations specifying the coordinate system. These in turn specify the four arbitrary functions appearing in the solutions of the field equations.

2. No coordinate conditions are added. The general solutions contain four arbitrary functions. If necessary, we may specify them, but we are not restricted in our choice by any added coordinate conditions.

3. All coordinate systems are permissible if they satisfy one condition: the world is Euclidean and the coordinate system becomes Cartesian at infinity.

As to the first choice: it is represented by Fock and his school. The coordinate condition added to the field equations is the "harmonic" one:

$$(\sqrt{-g}\, g^{\mu\nu})_{|\nu} = 0. \tag{1.9.9}$$

Obviously such a condition, if used exclusively, changes the spirit of G. R. T. since it distinguishes a group of coordinate systems. Indeed the harmonic coordinate condition gives especially simple solutions in the case of linearized gravitational equations in empty space. However, such a case of solutions of linearized equations seems to be of little physical interest. Otherwise the harmonic coordinate system is rarely used. For example, it is not used in the classical Schwarzschild solution, and, as we shall show in the next chapter, it has little to do with the problem of motion. Therefore, from the practical point of view its advantage is very small, if any.

Yet from the theoretical point of view, if the harmonic condition should always accompany the field equations, this seems to us to be a step backwards from the ideas of G. R. T. The ideas of G. R. T. were, if not deduced, then made plausible by the simple and very suggestive ideal experiment concerning the falling lift. True enough, this showed the equivalence of two systems based on the equality of inertial and gravitational mass, only in a restricted time-space region. Yet it was important for the development of G. R. T. that two systems, though not moving uniformly relatively to each other,

are equivalent in a certain sense. The observer outside the lift can say that the lift moves with accelerated motion in his gravitational field. The observer inside the lift may claim that his system is an inertial system towards which the earth moves with accelerated motion. All this reasoning becomes meaningless if we admit only harmonic coordinate systems.

Now, about the second possibility. We cannot describe motion without specifying a coordinate system. We must have one in order to check the results of our calculations with experience. But to bring in any special coordinate system is a violation of the ideas of G. R. T.

What remains, therefore, is the third possibility: all coordinate systems are allowed, as long as the field becomes Euclidian at infinity and the coordinate system Cartesian. To make our point clear, let us consider the simple case of the planetary motion around a very large sun. Let us take a plane at infinity, or so far from the sun that we can neglect its gravitational field. There, at infinity, there is sense in talking about planes, perfect clocks, and rigid rods. Every event in the world is mirrored uniquely on the plane by a light-ray emitted from this event and reaching the plane perpendicularly to it. Therefore, to every event there correspond four numbers T, X^k in the infinitely removed plane. Thus the motion of a planet has its image on such a plane. If the motion is periodic, then its image forms a closed line. Now we can turn the plane until the area circumscribed by the image of the moving planet becomes a maximum. Such mirror motion, in such an infinitely removed plane, is objective in this sense: for its description no further specification of the coordinate system is necessary. (We may regard such mirror motion as objective only if the original motion was also in a "plane"; that is if the observer at infinity does not find any Doppler effect.)

Thus we see from our example, that, without referring to any particular coordinate system we may, at least in principle, describe the motion objectively. If we call the coordinates on the "parallel" infinite plane T, X^k then we have

$$X^k = X^k(T) \tag{1.9.10}$$

as the objective description of the motion. In an arbitrary coordinate system we have:

$$\xi^k = \xi^k(t) \tag{1.9.11}$$

where, theoretically at least, we have

$$T = t+\infty \tag{1.9.12}$$

since the light-rays take infinite time to reach the infinitely distant plane.

Now we can define a convenient coordinate system in which we describe the motion (1.9.11) mirrored on the infinitely distant plane. It will be such that the motion on it is almost objective; that is, the difference between ξ^k and X^k and between ΔT and Δt is below the experimental error. We shall return to this problem later, perfecting mathematically the ideas described here only generally.

10. ON THE METHOD OF SOLUTION OF THE FIELD EQUATIONS WITH THE HELP OF THE DIPOLE PROCEDURE

If we wish to solve the problem of motion for point particles and find the gravitational field, that is, if we wish to find $\xi^\alpha(\lambda)$ and $g_{\alpha\beta}(x)$, then we must deal with the following system of equations:

$$\mathfrak{G}^{\alpha\beta} = -\frac{8\pi k}{c^2}\sum_{A=1}^{N} \overset{A}{m}_{(0)} \int\limits_{-\infty}^{\infty} ds_A\, \delta_{(4)}(x-\overset{A}{\xi})\frac{d\overset{A}{\xi}{}^\alpha}{ds_A}\frac{d\overset{A}{\xi}{}^\beta}{ds_A}, \tag{1.10.1}$$

$$\overset{A}{\Omega}{}^\alpha(s_A) = \frac{d^2\overset{A}{\xi}{}^\alpha}{ds_A{}^2} + \overset{A}{\overbrace{\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}}}\frac{d\overset{A}{\xi}{}^\mu}{ds_A}\frac{d\overset{A}{\xi}{}^\nu}{ds_A} = 0, \tag{1.10.2}$$

$$ds_A = (\overset{A}{\overbrace{g_{\alpha\beta}}}\,d\overset{A}{\xi}{}^\alpha d\overset{A}{\xi}{}^\beta)^{1/2}. \tag{1.10.3}$$

Equations (1.10.1) are those of the field; they determine $g_{\alpha\beta}$ as functions of the world-points.

Equations (1.10.2) are those of the motion; they determine $\overset{A}{\xi}{}^{\alpha}$ as functions of s_A, or $\overset{A}{\xi}{}^{k}$ as functions of x^0, if we introduce into them the corresponding change of the parameter.

But we also know that the two systems of equations are not independent of each other. Because of the Bianchi identities $\mathfrak{G}^{\alpha\beta}{}_{;\beta} = 0$ we have, as was shown before:

$$\left(\sum_{A=1}^{N} \overset{A}{m}_{(0)} \int_{-\infty}^{\infty} ds_A\, \delta_{(4)}(x - \overset{A}{\xi}) \frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A} \frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}\right)_{;\beta}$$

$$\equiv \sum_{A=1}^{N} \overset{A}{m}_{(0)} \int_{-\infty}^{\infty} ds_A\, \delta_{(4)}(x - \overset{A}{\xi}) \overset{A}{\Omega}{}^{\alpha}(s_A). \qquad (1.10.4)$$

Therefore, we repeat: equations (1.10.2) are a consequence of the field equations forming the condition for their integrability.

Thus, we cannot solve the field equations with arbitrary motion. That is: we cannot find the g's as functionals of arbitrary motion of the sources appearing on the right-hand side of the field equations (1.10.1). Such a solution will exist only if the motion satisfies the equations of motion, that is (1.10.2).

Therefore, the usual procedure of solving the field equations by assuming arbitrary motion, a useful procedure in all linear field theories, is of no avail here.

How can we avoid this difficulty? We have said before that there are two ways of doing so; firstly: the dipole procedure, which is of principial rather than of practical value, and which we shall describe presently; secondly: the approximation procedure with which we shall deal in Chapters II and III.

Modifying Einstein's equations we assume that besides the moving point particles the sources of the field consist of an additional, auxiliary field $\mathfrak{D}^{\alpha}$. This field has to be chosen in such a way as to make the motion of particles arbitrary. Thus we shall be able to solve the non-Einstein equations in which the artificial field appears devoid of any physical meaning; we shall be able to solve

these generalized non-Einstein equations for arbitrary motion. But in the case in which the motion is the right relativistic motion, then the additional $\mathfrak{D}^{\alpha}$ field vanishes. Or, conversely: if the additional field $\mathfrak{D}^{\alpha}$ vanishes, then the equations become the Einstein equations and the motion becomes the right motion, satisfying equations (1.10.2).

We call the additional, artificial field $\mathfrak{D}^{\alpha}$ the field of dipole potentials.

Let us, therefore, consider the field equations of General Relativity Theory with some additional tensor

$$\mathfrak{G}^{\alpha\beta} = -\frac{8\pi k}{c^4}(\mathfrak{T}^{\alpha\beta}+\mathfrak{S}^{\alpha\beta}) = -\frac{8\pi k}{c^4}\mathfrak{C}^{\alpha\beta}, \tag{1.10.5}$$

where, we assume, as in Section 8:

$$\mathfrak{T}^{\alpha\beta} = \sum_A \overset{A}{m}_{(0)} c^2 \int\limits_{-\infty}^{\infty} ds_A\, \delta_{(4)} \frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}, \qquad \mathfrak{S}^{\alpha\beta} = \mathfrak{S}^{\alpha\beta}(D, g, \xi'). \tag{1.10.6}$$

Then, the equations of motion are:

$$\mathfrak{C}^{\alpha\beta}{}_{;\beta} = \mathfrak{T}^{\alpha\beta}{}_{;\beta}+\mathfrak{S}^{\alpha\beta}{}_{;\beta} = 0. \tag{1.10.7}$$

We ask: is it possible to find the $\mathfrak{S}^{\alpha\beta}$ and the field equations for the D vector in such a way that the equations of motion would be fulfilled automatically for any arbitrary motion? Indeed, this is possible only if $\mathfrak{C}^{\alpha\beta}{}_{;\beta}$ becomes zero as a consequence of the field equations for $\mathfrak{D}^{\alpha}$. This is precisely the case if the field equations for $\mathfrak{D}^{\alpha}$ are:

$$\left.\begin{aligned} \mathfrak{D}^{\alpha;\beta}{}_{;\beta} - R^{\alpha\beta}\mathfrak{D}_{\beta} &= -\sum_A \overset{A}{m}_{(0)} c^2 \int\limits_{-\infty}^{\infty} ds_A\, \delta_{(4)}(x-\overset{A}{\xi})\overset{A}{\Omega}{}^{\alpha}(s_A), \\ \overset{A}{\Omega}{}^{\alpha}(s_A) &= \frac{d^2\overset{A}{\xi}{}^{\alpha}}{ds_A^2} + \overset{A}{\overbrace{\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}}}\frac{d\overset{A}{\xi}{}^{\mu}}{ds_A}\frac{d\overset{A}{\xi}{}^{\nu}}{ds_A} \end{aligned}\right\} \tag{1.10.8}$$

and

$$\mathfrak{S}^{\alpha\beta} = \mathfrak{D}^{\alpha;\beta} + \mathfrak{D}^{\beta;\alpha} - g^{\alpha\beta}\mathfrak{D}^{\varrho}{}_{;\varrho}. \tag{1.10.9}$$

Indeed, because of Section 6, we can rewrite equation (1.10.8) in the form:

$$\mathfrak{D}^{\alpha;\beta}{}_{;\beta} - R^{\alpha\beta}\mathfrak{D}_{\beta} = -\mathfrak{T}^{\alpha\beta}{}_{;\beta}. \tag{1.10.10}$$

According to the known rules of tensor calculus, we have

$$\begin{aligned}\mathfrak{E}^{\alpha\beta}{}_{;\beta} &= \mathfrak{D}^{\alpha;\beta}{}_{;\beta} + \mathfrak{D}^{\beta;\alpha}{}_{;\beta} - \mathfrak{D}^{\varrho}{}_{;\varrho}{}^{;\alpha} + \mathfrak{T}^{\alpha\beta}{}_{;\beta} \\ &= \mathfrak{D}^{\alpha;\beta}{}_{;\beta} - R^{\alpha\beta}\mathfrak{D}_{\beta} + \mathfrak{T}^{\alpha\beta}{}_{;\beta} = 0,\end{aligned} \tag{1.10.11}$$

which is identical with the field equations (1.10.10).

What are the characteristic features of this theory?

Firstly: in this theory the motion of particles is arbitrary. Indeed, using Bianchi identities in (1.10.5), we obtain the field equations for the additional field $\mathfrak{D}^{\alpha}$, which are satisfied anyhow.

Thus the system of equations (1.10.5-8) has a solution for arbitrary motion and we can find from it:

$$g_{\alpha\beta} = g_{\alpha\beta}(x)[\overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{N}{\xi}], \tag{1.10.12}$$

$$\mathfrak{D}^{\alpha} = \mathfrak{D}^{\alpha}(x)[\overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{N}{\xi}]. \tag{1.10.13}$$

Here the square brackets, as usual, denote functional dependence

Secondly: let us put $\mathfrak{D}^{\alpha} = 0$ into equations (1.10.5-8); then those generalized field equations become Einstein's equations, and the equations of motion (1.10.2) must be satisfied. Thus, at the end of our procedure we may return to Einstein's field equations, finding the motion from the condition that the additional dipole field vanishes.

Thirdly: let us assume that the field equations $\Omega^{\alpha} = 0$ are fulfilled; that means that we have chosen the motion, which is arbitrary in the generalized equations, so that it is the right motion according to G. R. T. Does it follow from this choice of the motion that the field $\mathfrak{D}^{\alpha}$ vanishes? This is not necessarily so, because the field equations for $\mathfrak{D}^{\alpha}$ are then:

$$\mathfrak{D}^{\alpha;\beta}{}_{;\beta} - R^{\alpha\beta}\mathfrak{D}_{\beta} = 0. \tag{1.10.8}$$

These homogeneous equations may have solutions different from

zero. Only if we also assume that as a solution of $\mathfrak{D}^a$ for the case $\overset{A}{\Omega}{}^a = 0$, we take the solution $\mathfrak{D}^a \equiv 0$, do we have:

$$\overset{A}{\Omega}{}^a = 0 \rightleftarrows \mathfrak{D}_a(x)[\overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{N}{\xi}] \equiv 0. \tag{1.10.14}$$

Let us assume that we have solved the generalized system of equations for $g_{\alpha\beta}(x)$. Let us put these g's into $\overset{A}{\Omega}{}^a(s_A)$. If we wish, since the motion is arbitrary, we may assume the equations of motion

$$\overset{A}{\Omega}{}^a = 0. \tag{1.10.15}$$

Since in this case we have agreed to admit only the solution $\mathfrak{D}^a = 0$, then the field equations become the Einstein equations.

Thus we have formulated the general theory, according to which we can find the equations of motion of the third kind. There will be more on this subject in the next Chapters.

Why do we call this additional field the field of dipole potential? The answer is simple. Looking at the field equations (1.10.8) for $\mathfrak{D}^a$, we see that its solution will have the form:

$$\mathfrak{D}^a \sim \sum_A \frac{\overset{A}{\Omega}{}^a}{\left|\mathbf{x} - \overset{A}{\xi}\right|}. \tag{1.10.16}$$

Therefore, in the additional field appearing in (1.10.5) for $g_{\alpha\beta}$ we have derivatives of $\mathfrak{D}^a$ — expressions which behave as though they represented "gravitational dipoles". But gravitational dipoles do not exist in reality. By getting rid of this artificial field we return to Einstein's equations and the proper equations of motion.

CHAPTER II

THE APPROXIMATION METHOD AND THE EQUATIONS OF MOTION

1. GENERAL REMARKS ON THE APPROXIMATION METHOD

In the first chapter we gave the rigorous theory of the equations of motion in G. R. T.; we discussed the connection between these equations and those of the gravitational field. But what we said on this subject was rather formal. We did not consider any physical problems that could be compared with observation, at least in principle.

Because of the non-linearity of the equations we are not able, for example, to give a rigorous solution of the two-body problem which, in Newtonian theory, is as easy as the one-body problem. We cannot find the field; therefore we cannot find the motion. All that we can do is to use an approximation method which will allow us to find the field and the motion approximately. Such a method is more than adequate from a practical point of view. It will lead us to new effects which can barely be confirmed even by the most precise observations.

In the case of non-linear equations the only method of approximation used is that of the small parameter. Roughly speaking, the method consists in developing all quantities which appear in these equations in power series of this small parameter, then solving successively the equations formed by the coefficients of those series.

What quantity should we choose as the small parameter in G. R. T.? It would seem that a good choice would be the gravita-

tional constant k which plays the role of a coupling constant between the geometry described by the Einstein tensor $G^{\mu\nu}$ and the physics described by the energy-momentum tensor $T^{\mu\nu}$. However, this is not so. We demand that our parameter give us Newtonian motion in the first approximation, post-Newtonian motion in the next. However, using k as a parameter, we obtain uniform motion in the first approximation, and in the next — Newtonian motion plus forces depending on the velocity of light. Such a method is certainly unsuited to the description of planetary motion, where the velocities are much smaller than the velocity of light.

Thus another possibility, the one that we shall choose here, is to take as the small parameter

$$\lambda = \frac{1}{c}; \tag{2.1.1}$$

$c =$ velocity of light.

This method turns out to be very effective for the investigation of the problem of motion in G. R. T.

Up to the present day the Newtonian theory, in which the velocity of light does not appear at all, forms the basis of astronomy. In comparison with this theory, G. R. T. is much more complicated from the technical point of view. Yet from the principal point of view, it is simpler. It does not require the strange concepts of action at a distance, or an inertial coordinate system, and the equality of inertial and gravitational mass does not appear in it as an artificial assumption. Yet the practical usefulness of G. R. T. demands that the following two conditions be fulfilled:

1. Newtonian theory follows from G. R. T. as a special limiting case.

2. G. R. T. can, without introducing any arbitrary constants, predict new events which do not agree with Newtonian theory.

From what limiting case of G. R. T. can we obtain Newtonian theory? The answer is obvious. Newtonian theory, which is not a proper field theory and in which the constant c does not appear, can only follow from G. R. T. in the limiting case $c \to \infty$; or in

other words, "action at a distance" means a propagation with infinite speed.

Thus we expect that for

$$\lambda = \frac{1}{c} \to 0 \tag{2.1.2}$$

the equations of G. R. T. become those of Newtonian theory. In practice, we can regard c as infinitely great if v/c (where v is the characteristic speed of the body) is much smaller than one. On the other hand, if we do not neglect v/c, we may be led to new phenomena outside the frame of Newtonian theory.

We believe that these remarks justify the choice of $\lambda = 1/c$ as the parameter of our approximation procedure. Thus we assume that all our quantities are analytical in λ. That is, if f is such a quantity, we shall write:

$$f = \underset{0}{f} + \underset{1}{f} + \underset{2}{f} + \underset{3}{f} + \dots \tag{2.1.3}$$

We denote the order of each of these quantities by a subscript, that is:

$$f = \sum_{n=0}^{\infty} \lambda^n f_n, \qquad \underset{n}{f} = \lambda^n f_n. \tag{2.1.4}$$

Besides the obvious rules (e. g. that the product $\underset{k}{\varphi}\underset{l}{\psi}$ is of the order $k+l$) there is one which is less obvious but essential to our later argument. It is connected with the differentiation of f with respect to time t, where

$$t = \frac{x^0}{c} = \lambda x^0. \tag{2.1.5}$$

We assume t to be of the order zero in λ. Therefore:

$$\underset{0}{f}_{|0} = \frac{\partial f}{\partial x^0} = \frac{1}{c}\frac{\partial f}{\partial t} = \lambda \frac{\partial f}{\partial t} \tag{2.1.6}$$

must be of the order one. Or generally, we could write:

$$\underset{l}{f}_{\underset{1}{|0}} = \underset{\underbrace{\ }_{l+1}}{f_{|0}}. \tag{2.1.7}$$

In words: differentiation of f with respect to x^0 raises the order of f by one; we shall omit the subscript 1 under a zero denoting differentiation with respect to x^0 and shall write simply

$$\frac{\partial \underset{l}{f}}{\partial x^0} = \underset{l+1}{f}_{|0}.$$

Differentiation with respect to x^k does not change the order of f. This means that all quantities are assumed to change slowly with x^0.

2. ON THE DEVELOPMENT OF THE METRIC FIELD

We shall deal here solely with isolated material systems. Far from any matter, we shall have a vanishing curvature of the time-space continuum. Thus we can assume a flat time-space at infinity in which we can still fix the coordinate system in an arbitrary way. However, for the sake of simplicity, we shall assume a Cartesian coordinate system there. Thus, as far as measurements are concerned, there are the same conditions at infinity as in an inertial coordinate system, and as far as the metric is concerned we have for $r^2 = x^s x^s$:

$$\lim_{r\to\infty} g_{\alpha\beta}(x) = \eta_{\alpha\beta}, \quad (\eta_{00} = 1, \eta_{0k} = 0, \eta_{ik} = -\delta_{ik}). \tag{2.2.1}$$

The transition from the Riemannian metric $g_{\alpha\beta}$ to the Cartesian or Galilean metric $\eta_{\alpha\beta}$ at infinity plays an essential role in the theory of measurement as previously outlined (I. Section 9). We shall return to this subject later (V. Section 1).

We shall now write the Riemannian metric in the form:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{2.2.2}$$

where the $h_{\alpha\beta}$ go to zero as the distance $r = (x^s x^s)^{1/2}$ goes to infinity. Now we wish to develop $h_{\alpha\beta}$ into a power series in λ. Since for $c \to \infty$ we have $h_{\alpha\beta} \to 0$, we must start $h_{\alpha\beta}$ with at least the order one, that is

$$h_{\alpha\beta} = \underset{1}{h}_{\alpha\beta} + \underset{2}{h}_{\alpha\beta} + \underset{3}{h}_{\alpha\beta} + \dots \tag{2.2.3}$$

But by examining the structure of the field equations of G. R. T. more deeply, we can say more. We can show that

$$\underset{1}{h}_{00} = 0, \quad \underset{1}{h}_{0m} = 0. \tag{2.2.4}$$

(Later we shall learn to restrict $h_{\alpha\beta}$ still further).

Indeed, we can show the validity of the last equation by comparing the equations of motion for a test particle in Newtonian theory with those in G. R. T. In the Newtonian theory we have for a test particle of mass Δm under the influence of a gravitational field φ:

$$\delta(-\Delta m)\int_{t_1}^{t_2} dt(c^2 - \tfrac{1}{2}\dot{\xi}^s\dot{\xi}^s + \varphi) = 0, \quad \dot{\xi}^s = \frac{d\xi^s}{dt}. \tag{2.2.5a}$$

The equation of motion for a test particle in G. R. T. is:

$$(-\Delta m)\,\delta\int_{t_1}^{t_2} \frac{ds}{dt}\,dt = 0, \quad ds^2 = (\eta_{\alpha\beta} + h_{\alpha\beta})\,d\xi^\alpha d\xi^\beta. \tag{2.2.5b}$$

Developing the square root with respect to λ we obtain from the last equation:

$$(-\Delta m)\,\delta\int_{t_1}^{t_2} dt(c^2 - \tfrac{1}{2}\dot{\xi}^s\dot{\xi}^s + \tfrac{1}{2}c^2h_{00} + ch_{0n}\dot{\xi}^n + \tfrac{1}{2}h_{mn}\dot{\xi}^m\dot{\xi}^n) = 0. \tag{2.2.6}$$

Therefore:

$\tfrac{1}{2}c^2h_{00} \to \varphi$ or like λ^0;

$ch_{0n}\dot{\xi}^n \to$ at least like $1/c$, or like λ;

$h_{mn}\dot{\xi}^m\dot{\xi}^n \to$ at least like $1/c$, or like λ.

The last correspondence proves the validity of (2.2.4). Therefore we have:

$$\left.\begin{aligned} h_{00} &= \underset{2}{h_{00}} + \underset{3}{h_{00}} + \underset{4}{h_{00}} + \dots, \\ h_{0n} &= \underset{2}{h_{0n}} + \underset{3}{h_{0n}} + \underset{4}{h_{0n}} + \dots, \\ h_{mn} &= \underset{1}{h_{mn}} + \underset{2}{h_{mn}} + \underset{3}{h_{mn}} + \dots, \end{aligned}\right\} \tag{2.2.7}$$

as a condition for the consistency of Newtonian Theory and G. R. T.

3. ON THE THREE LEVELS OF OUR REASONING

In principle, we picture material bodies whose motion we wish to describe, as singularities of the gravitational field. This leads to the use of the good δ functions, which help us to avoid the process of renormalization. We shall say that if such point particles are assumed, we reason on the level of field singularities.

To deepen the method of using the δ function we shall have to ascend to the level of continuous distribution. That is, we shall assume continuous fluid matter. This problem is of some interest in itself. We may, and shall, descend by a limiting process from this level to that of field singularities.

Finally we shall often use the still more general level of an arbitrary energy-momentum tensor, especially for discussions in which its specific form is of little importance.

Thus on the level of field singularities, we have:

$$\mathfrak{G}^{\mu\nu} = -8\pi\mathcal{T}^{\mu\nu} = -\frac{8\pi k}{c^2}\sum_{A=1}^{N}\overset{A}{m}_{(0)}\int_{-\infty}^{\infty} ds_A\,\delta_{(4)}(x-\overset{A}{\xi})\frac{d\overset{A}{\xi^{\mu}}}{ds_A}\frac{d\overset{A}{\xi^{\nu}}}{ds_A}. \tag{2.3.1}$$

We recall that $\overset{A}{\xi^{\nu}}(s_A)$ is the world-line of the A'th singularity; $\overset{A}{m}_{(0)}$ is its rest mass; because of the use of the good δ functions, ds_A is the regularized time-space interval of the A'th particle.

On the level of continuous distributions we have:

$$\mathfrak{G}^{\mu\nu} = -8\pi\mathcal{T}^{\mu\nu} = -\frac{8\pi k}{c^2}\sqrt{-g}\left[\left(\varrho+\frac{\varrho}{c^2}\int_0^p\frac{dp}{\varrho}\right)u^{\mu}u^{\nu}-\frac{p}{c^2}g^{\mu\nu}\right]. \tag{2.3.2}$$

Here the density ϱ is a function of the pressure p; $u^{\alpha}(x)$ is the Eulerian field of the velocity vector. These quantities satisfy the conditions expressing the conservation law of matter:

$$g_{\alpha\beta}u^{\alpha}u^{\beta} = 1, \qquad (\varrho u^{\alpha})_{;\alpha} = 0 \tag{2.3.3}$$

(if we assume the first of these-equations then the second one can be deduced from the conservation law $\mathcal{T}^{\mu\nu}{}_{;\nu} = 0$ by contracting it with u_{μ}).

On the level of an arbitrary energy-momentum tensor we have

$$\mathfrak{G}^{\mu\nu} = -\frac{8\pi k}{c^4}\mathfrak{T}^{\mu\nu} = -8\pi\mathcal{T}^{\mu\nu}(A^{\alpha}, g_{\varrho\sigma}). \tag{2.3.4}$$

Here A^{α} stands for the dynamical quantities describing the physical field; like the $\dot{\xi}^{\alpha}$ on the first level and ϱ, p, u^{α} on the second. We shall assume, as on the first two levels, that $\mathcal{T}^{\mu\nu}(A, g)$ vanishes outside a sufficiently large volume.

We ask: with what orders should we start the development of the $\mathcal{T}^{\mu\nu}$ tensor? We shall answer the question first on the level of field singularities. There we have (see I, 4)

$$\mathcal{T}^{\mu\nu} = \sum_{A=1}^{N} \overset{A}{\mu}\,\delta(\mathbf{x}-\overset{A}{\boldsymbol{\xi}})\,\overset{A}{\xi}{}^{\mu}{}_{|0}\,\overset{A}{\xi}{}^{\nu}{}_{|0}, \qquad \xi^0 = x^0, \qquad \xi^0{}_{|0} = 1,$$

$$\overset{A}{\mu} = \frac{k\overset{A}{m}_{(0)}}{c^2}\frac{dx^0}{ds_A}. \tag{2.3.5a}$$

We shall assume ξ^k to begin with the order zero:

$$\xi^k = \underset{0}{\xi^k} + \underset{1}{\xi^k} + \underset{2}{\xi^k} + \ldots \tag{2.3.5b}$$

The function $\overset{A}{\mu}$ starts with the order two. This we see from (2.3.5a). We also see it from the Newtonian equations of motion. There we have (in the case of a two-body problem)

$$\overset{A}{\mu}\,\underset{2}{\overset{A}{\xi}{}^{k}}{}_{|00} = \overset{A}{\mu}\overset{B}{\mu}\frac{\partial}{\partial \overset{A}{\xi}{}^{k}}|\overset{A}{\boldsymbol{\xi}} - \overset{B}{\boldsymbol{\xi}}|^{-1}. \tag{2.3.6}$$

Since the lowest order of both sides of this equation must be the same, we conclude:

$$\mu = \underset{2}{\mu} + \underset{3}{\mu} + \underset{4}{\mu} + \ldots \tag{2.3.7}$$

Therefore we see:

$$\left.\begin{aligned} \mathcal{T}^{00} &= \underset{2}{\mathcal{T}}^{00} + \underset{3}{\mathcal{T}}^{00} + \underset{4}{\mathcal{T}}^{00} + \ldots, \\ \mathcal{T}^{0n} &= \qquad\quad \underset{3}{\mathcal{T}}^{0n} + \underset{4}{\mathcal{T}}^{0n} + \ldots, \\ \mathcal{T}^{mn} &= \qquad\qquad\qquad \underset{4}{\mathcal{T}}^{mn} + \ldots. \end{aligned}\right\} \tag{2.3.8}$$

In a similar way we obtain the same result on the level of continuous distribution, and we assume that on the level of the arbitrary energy-momentum tensor we have, exactly as in (2.3.8):

$$\mathcal{T}^{00} = O(\lambda^2), \qquad \mathcal{T}^{0n} = O(\lambda^3), \qquad \mathcal{T}^{mn} = O(\lambda^4). \tag{2.3.9}$$

Since $\mathcal{T}^{\alpha\beta}$ is a function of A^α and $g_{\varrho\sigma}$, the last equation determines the lowest order in λ of A^α.

4. THE APPROXIMATION METHOD AND THE COORDINATE SYSTEM

What is the relation of a coordinate transformation to the approximation method? That is: what conditions must the coordinate transformation fulfil so as not to upset the properties of the metric field, either at infinity, or in the order with which its development starts? We write the transformation to the starred coordinate system:

$$x^{*\alpha} = x^\alpha + a^\alpha(x, \lambda), \tag{2.4.1}$$

and we assume that this transformation becomes an identity for $r \to \infty$, that is

$$\lim_{r\to\infty} a^\alpha(x, \lambda) = 0. \tag{2.4.2}$$

Then from the formula:

$$g_{\alpha\beta} = \overset{*}{g}_{\mu\nu} x^{*\mu}{}_{|\alpha} x^{*\nu}{}_{|\beta} \tag{2.4.3}$$

we deduce, neglecting the products $a \cdot h$ and $a \cdot a$:

$$\left.\begin{aligned} h_{00} &= h^*_{00} + 2a^0{}_{|0}, \\ h_{0n} &= h^*_{0n} + a^0{}_{|n} - a^n{}_{|0}, \\ h_{mn} &= h^*_{mn} - a^n{}_{|m} - a^m{}_{|n}. \end{aligned}\right\} \tag{2.4.4}$$

From the last equation, the answer to our question follows immediately. Since, according to (2.2.7), the h's start with $\underset{2}{h_{00}}, \underset{2}{h_{0n}}, \underset{1}{h_{mn}}$, we see that, if we wish the h^*'s to start the same way, we must have:

$$\left.\begin{aligned} a^0 &= \underset{2}{a^0} + \underset{3}{a^0} + \underset{4}{a^0} + \ldots \\ a^n &= \underset{1}{a^n} + \underset{2}{a^n} + \underset{3}{a^n} + \ldots \end{aligned}\right\} \tag{2.4.5}$$

We can easily generalize this result. Let us take for (2.4.1)

$$x^{*0} = x^0 + \underset{l+1}{a^0}, \qquad x^{*k} = x^k + \underset{l}{a^k} \tag{2.4.6}$$

then the first $h_{\alpha\beta}$ affected by such a transformation are:

$$\underset{l}{h_{mn}}, \underset{l+1}{h_{0n}}, \underset{l+2}{h_{00}}. \tag{2.4.7}$$

Indeed we have:

$$\left.\begin{aligned} \underset{l}{h_{mn}} &= \underset{l}{h^*{}_{mn}} - \underset{l}{a^m}{}_{|n} - \underset{l}{a^n}{}_{|m}, \\ \underset{l+1}{h_{0n}} &= \underset{l+1}{h^*{}_{0n}} + \underset{l+1}{a^0}{}_{|n} - \underset{l+1}{a^n}{}_{|0}, \\ \underset{l+2}{h_{00}} &= \underset{l+2}{h^*{}_{00}} + 2\underset{l+2}{a^0}{}_{|0}. \end{aligned}\right\} \tag{2.4.8}$$

Let us recall one of the most important theorems of Riemannian geometry, which concerns the necessary and sufficient conditions for a space to be pseudo-Euclidean. Thus we ask: when can a coordinate transformation change the metric form

$$g_{\alpha\beta}\,dx^\alpha\,dx^\beta \quad \text{into} \quad \eta_{\alpha\beta}\,dx^\alpha\,dx^\beta\,? \tag{2.4.9}$$

The necessary and sufficient condition for this to happen is known to be:

$$R_{\mu\nu\varrho\sigma} = 0 \tag{2.4.10}$$

that is, the vanishing of the great Riemannian tensor. The proof that this is a necessary condition is trivial and can be dismissed with the remark that $g_{\alpha\beta} = \eta_{\alpha\beta}$ annihilates the great Riemannian tensor. The proof that it is a sufficient condition is much more troublesome and many different proofs can be found in books. At least one of them is based on the fact that (2.4.10) forms the integrability condition for the existence of four transformation functions by which $g_{\alpha\beta}$ is changed into $\eta_{\alpha\beta}$.

Now, in connection with our approximation procedure we may ask a similar question: what are the necessary and sufficient conditions for the existence of a transformation $\underset{l+1}{a^0}$, $\underset{l}{a^k}$ that turns the ten functions

$$\underset{l}{h_{mn}}, \underset{l+1}{h_{0n}}, \underset{l+2}{h_{00}} \tag{2.4.11}$$

into zeros?

Let us introduce the linearized curvature tensor:

$$S_{\mu\alpha,\nu\beta} = \tfrac{1}{2}(h_{\alpha\beta|\mu\nu} + h_{\mu\nu|\alpha\beta} - h_{\mu\beta|\nu\alpha} - h_{\alpha\nu|\mu\beta}). \tag{2.4.12}$$

Then the necessary and sufficient condition for a coordinate transformation to annihilate $\underset{l}{h_{mn}}, \underset{l+1}{h_{0n}}, \underset{l+2}{h_{00}}$ is:

$$\underset{l}{S_{ma,nb}} = 0, \qquad \underset{l+1}{S_{m0,nb}} = 0, \qquad \underset{l+2}{S_{m0,0b}} = 0. \tag{2.4.13}$$

The proof that (2.4.13) is necessary is trivial. The proof that this condition is sufficient is more complicated as it is similar to the more general case where no approximation procedure appears. We shall omit it here.

Having the $h_{\alpha\beta}$, we can define the $h^{\alpha\beta}$ through the equation

$$g^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta}, \qquad h^{\alpha\beta} = \sum_{l=1}^{\infty} \underset{l}{h^{\alpha\beta}}. \tag{2.4.14}$$

We can find the successive values of $\underset{l}{h}^{\alpha\beta}$ from those of $\underset{k}{h}_{\alpha\beta}$ $(k \leqslant l)$ since

$$g^{\alpha\beta}g_{\beta\gamma} = \delta^{\alpha}_{\gamma}.$$

Thus we find that the $h^{\alpha\beta}$ start with

$$\underset{2}{h}^{00} = -\underset{2}{h}_{00},\ \underset{2}{h}^{0m} = \underset{2}{h}_{0m},\ \underset{1}{h}^{mn} = -\underset{1}{h}_{mn},\ \underset{3}{h}^{00} = -\underset{3}{h}_{00},$$

$$\underset{4}{h}^{00} = (\underset{2}{h}_{00})^2 - \underset{4}{h}_{00}. \tag{2.4.15}$$

The above formulas will be sufficient for all our practical calculations. Merely for the sake of completeness we quote the general formula:

$$\underset{l}{h}^{\alpha\beta} = \sum (-1)^p \eta^{\alpha\varrho_1} \underset{k_1}{h}_{\varrho_1\varrho_2} \eta^{\varrho_2\varrho_3} \underset{k_2}{h}_{\varrho_3\varrho_4} \ldots \underset{k_p}{h}_{\varrho_{2p-1}\varrho_{2p}} \eta^{\varrho_{2p}\beta}, \tag{2.4.16}$$

where the sum is to be extended over all combinations (of positive integers) satisfying $k_1 + k_2 + \ldots + k_p = l$.

5. THE APPROXIMATION METHOD AND THE FIELD EQUATIONS

We shall now introduce the general scheme of dealing with the equations of G. R. T. by the approximation method. It will be slightly more convenient — at least in the general scheme — to treat as our fundamental quantities, not the h's, but the γ's. Let us therefore start with the definition of the γ's. We write:

$$\sqrt{-g}\, g^{\alpha\beta} = \mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + \gamma^{\alpha\beta} \tag{2.5.1}$$

which equation gives the definition of the γ's. They will be regarded as our principal expressions characterizing the metric field; from them all others can be derived:

$$g = |g_{\alpha\beta}| = |g^{\alpha\beta}|^{-1} = |\eta^{\alpha\beta} + \gamma^{\alpha\beta}|^{-1} g^2 \tag{2.5.2}$$

from which follows

$$g = |\eta^{\mu\nu} + \gamma^{\mu\nu}|, \tag{2.5.3}$$

therefore

$$\sqrt{-g} = 1 + \tfrac{1}{2}\eta_{\alpha\beta}\gamma^{\alpha\beta} + \text{non-linear expressions.} \tag{2.5.4}$$

From the last equation and the definition of $h^{\alpha\beta}$ follows:

$$h^{\alpha\beta} = \gamma^{\alpha\beta} - \tfrac{1}{2}\eta^{\alpha\beta}\eta_{\varrho\sigma}\gamma^{\varrho\sigma} + \text{non-linear expressions.} \tag{2.5.5}$$

We see, because of (2.4.15):

$$\gamma^{00} = \underset{1}{\gamma^{00}} + \underset{2}{\gamma^{00}} + \underset{3}{\gamma^{00}} + \dots \tag{2.5.6a}$$

$$\gamma^{0m} = \underset{2}{\gamma^{0m}} + \underset{3}{\gamma^{0m}} + \dots \tag{2.5.6b}$$

$$\gamma^{mn} = \underset{1}{\gamma^{mn}} + \underset{2}{\gamma^{mn}} + \underset{3}{\gamma^{mn}} + \dots \tag{2.5.6c}$$

We shall now consider the fundamental equations of G. R. T.

$$\mathfrak{G}^{\mu\nu} + 8\pi\mathcal{T}^{\mu\nu} = 0. \tag{2.5.7}$$

We shall start by calculating explicitly the linear part of $\mathfrak{G}^{\mu\nu}$, that is the part that contains only the second derivatives of the γ's and those only linearly. The Einstein tensor density $\mathfrak{G}^{\mu\nu}$ can be put in the form

$$\mathfrak{G}^{\mu\nu} = \frac{1}{2\sqrt{-g}}(\mathfrak{g}^{\mu\alpha}\mathfrak{g}^{\nu\beta} - \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta})_{|\alpha\beta} + \mathfrak{G}^{*\mu\nu} \tag{2.5.8}$$

where $\mathfrak{G}^{*\mu\nu}$ is a function quadratic in the first derivatives.

Therefore, the linear part of $\mathfrak{G}^{\mu\nu}$ can be written as

$$L(\mathfrak{G}^{\mu\nu}) = K^{\mu\alpha\nu\beta}{}_{|\alpha\beta}, \tag{2.5.9}$$

where

$$K^{\mu\alpha\nu\beta} = K^{\mu\alpha,\nu\beta} = \tfrac{1}{2}(\eta^{\nu\alpha}\gamma^{\mu\beta} + \eta^{\mu\beta}\gamma^{\nu\alpha} - \eta^{\mu\nu}\gamma^{\alpha\beta} - \eta^{\alpha\beta}\gamma^{\mu\nu}). \tag{2.5.10}$$

It is interesting to note the symmetry properties of the expression $K^{\mu\alpha,\nu\beta}$. We see that $K^{\mu\alpha,\nu\beta}$ is skew-symmetric in the indices μ, α and skew-symmetric in the indices ν, β. We see that $K^{\mu\alpha,\nu\beta}$ is symmetric with respect to the simultaneous replacement of μ by ν

and α by β. Finally $K^{\mu\alpha,\nu\beta}$ satisfies the relation

$$K^{\mu\alpha\nu\beta}+K^{\mu\nu\beta\alpha}+K^{\mu\beta\alpha\nu} = 0. \tag{2.5.11}$$

Thus we see that the expression $K^{\mu\alpha,\nu\beta}$ has all the symmetry properties of the great Riemannian tensor.

We write the fundamental equations of G. R. T. in the form:

$$K^{\mu\alpha\nu\beta}{}_{|\alpha\beta}+N(\mathfrak{G}^{\mu\nu})+8\pi\mathcal{T}^{\mu\nu} = 0, \tag{2.5.12}$$

where the symbol $N(S)$ means all the non-linear terms in S.

Let us now discuss briefly the transformation properties of $\gamma^{\mu\nu}$. Since $\mathfrak{g}^{\mu\nu}$ is a tensor density, we have:

$$\mathfrak{g}^{*\mu\nu} = \mathfrak{g}^{\alpha\beta}x^{*\mu}{}_{|\alpha}x^{*\nu}{}_{|\beta}\det\frac{\partial x}{\partial x^*}.$$

Into this equation we introduce

$$x^{*\mu}{}_{|\alpha} = \delta^\mu_\alpha+a^\mu{}_{|\alpha}, \tag{2.5.13}$$

$$\det\frac{\partial x}{\partial x^*} = 1-a^\alpha{}_{|\alpha}. \tag{2.5.14}$$

Neglecting the products aa and $a\gamma$ we conclude:

$$\gamma^{*\mu\nu} = \gamma^{\mu\nu}+\eta^{\alpha\mu}a^\nu{}_{|\alpha}+\eta^{\alpha\nu}a^\mu{}_{|\alpha}-\eta^{\mu\nu}a^\alpha{}_{|\alpha}, \tag{2.5.15}$$

or explicitly:

$$\gamma^{*00} = \gamma^{00}+a^0{}_{|0}-a^s{}_{|s} \tag{2.5.16a}$$

$$\gamma^{*0n} = \gamma^{0n}-a^0{}_{|n}+a^n{}_{|0} \tag{2.5.16b}$$

$$\gamma^{*mn} = \gamma^{mn}-a^m{}_{|n}-a^n{}_{|m}+\delta^{mn}a^s{}_{|s}+\delta^{mn}a^0{}_{|0}. \tag{2.5.16c}$$

We shall not upset the orders with which we start the γ's, if we develop the a's in the same way as before, that is:

$$a^0 = \underset{2}{a^0}+\underset{3}{a^0}+\dots, \tag{2.5.17a}$$

$$a^m = \underset{1}{a^m}+\underset{2}{a^m}+\dots. \tag{2.5.17b}$$

After these introductory remarks we pass to the chief problem of this section which is to adapt the approximation procedure to the field equations of G. R. T.

We rewrite (2.5.12):

$$K^{0a,0b}{}_{|ab} + N(\mathfrak{G}^{00}) + 8\pi\mathfrak{T}^{00} = 0, \tag{2.5.18a}$$

$$K^{0a,nb}{}_{|ab} + K^{0a,n0}{}_{|a0} + N(\mathfrak{G}^{0n}) + 8\pi\mathfrak{T}^{0n} = 0, \tag{2.5.18b}$$

$$K^{ma,nb}{}_{|ab} + K^{m0,nb}{}_{|0b} + K^{ma,n0}{}_{|a0} + K^{m0,n0}{}_{|00} + N(\mathfrak{G}^{mn}) + 8\pi\mathfrak{T}^{mn} = 0. \tag{2.5.18c}$$

Writing (2.5.10) more explicitly we have:

$$K^{ma,nb} = \tfrac{1}{2}(-\delta^{na}\gamma^{mb} - \delta^{mb}\gamma^{na} + \delta^{mn}\gamma^{ab} + \delta^{ab}\gamma^{mn}), \tag{2.5.19a}$$

$$K^{0a,nb} = \tfrac{1}{2}(-\delta^{na}\gamma^{0b} + \delta^{ab}\gamma^{0n}), \tag{2.5.19b}$$

$$K^{0a,0b} = \tfrac{1}{2}(-\gamma^{ab} + \delta^{ab}\gamma^{00}). \tag{2.5.19c}$$

Thus $K^{ma,nb}$ and $K^{0a,0b}$ start with the order one and $K^{0a,nb}$ with the order two. But, since $\mathfrak{T}^{mn}$ starts with the order four, $\mathfrak{T}^{0n}$ with the order three, and $\mathfrak{T}^{00}$ with the order two, if we start this time from the bottom we have in the lowest order:

$$\underset{1}{K}{}^{ma,nb}{}_{|ab} = 0, \qquad \underset{1}{K}{}^{0a,0b}{}_{|ab} = 0, \qquad \underset{2}{K}{}^{0a,nb}{}_{|ab} = 0. \tag{2.5.20}$$

The general solution of the first of these equations gives

$$\underset{1}{\gamma}{}^{mn} = \underset{1}{b}{}^{m}{}_{|n} + \underset{1}{b}{}^{n}{}_{|m} - \delta^{mn}\underset{1}{b}{}^{s}{}_{|s}, \tag{2.5.21}$$

where the b's are arbitrary functions. Comparing this with (2.5.16c), we see that $\underset{1}{\gamma}{}^{mn}$ can be annihilated by a proper choice of the coordinate system. We assume this to be done, which leads to $\underset{1}{\gamma}{}^{mn} = 0$ and $K^{ma,nb}$ starting at least with the order two. Because of this result, we obtain for the differential equation, (since $\mathfrak{T}^{00}$ is of the order two):

$$\underset{1}{K}{}^{0a,0b}{}_{|ab} = \tfrac{1}{2}\underset{1}{\gamma}{}^{00}{}_{|ss} = 0. \tag{2.5.22}$$

Similarly, the third of the equations in (2.5.20) gives

$$\underset{2}{\gamma}^{0n} = \underset{2}{b}^{0}{}_{|n}. \tag{2.5.23}$$

This also shows that $\underset{2}{\gamma}^{0n}$ can be annihilated by a proper time transformation. Thus the result is: merely by restricting our coordinate system we can always choose

$$\underset{1}{\gamma}^{mn} = \underset{1}{\gamma}^{00} = \underset{2}{\gamma}^{0n} = 0. \tag{2.5.24}$$

Let us now go one step further, again starting from the bottom of (2.5.18). We find:

$$\underset{2}{K}^{ma,nb}{}_{|ab} = 0, \qquad \underset{2}{K}^{0a,0b}{}_{|ab} + 8\pi \underset{2}{\mathcal{T}}^{00} = 0. \tag{2.5.25}$$

Thus now, as before, we conclude:

$$\underset{2}{\gamma}^{mn} = 0. \tag{2.5.26}$$

But this is no longer true for $\underset{2}{\gamma}^{00}$. It must satisfy the equation

$$\tfrac{1}{2}\underset{2}{\gamma}^{00}{}_{|ss} + 8\pi \underset{2}{\mathcal{T}}^{00} = 0. \tag{2.5.27}$$

Therefore $\underset{2}{\gamma}^{00} \neq 0$. Now about $\underset{3}{\gamma}^{0n}$. Since the non-linear terms must be of at least order four, we may omit them. We then have:

$$\underset{3}{K}^{0a,nb}{}_{|ab} + \underset{2}{K}^{0a,n0}{}_{\underset{1}{|0a}} + 8\pi \underset{3}{\mathcal{T}}^{0n} = 0, \tag{2.5.28}$$

which also gives us a value for $\underset{3}{\gamma}^{0n}$ which is different from zero. Turning once more to (2.5.18c) we see that the first γ^{mn} which must be assumed different from zero is $\underset{4}{\gamma}^{mn}$, since $K^{m0,nb}{}_{|b0}$, $K^{m0,n0}{}_{|00}$, $N(\mathfrak{G}^{mn})$, and $\mathcal{T}^{mn}$ start with the order four. Thus in a properly chosen

coordinate system we may assume

$$\gamma^{00} = \underset{2}{\gamma^{00}} + \underset{3}{\gamma^{00}} + \underset{4}{\gamma^{00}} + \ldots, \tag{2.5.29a}$$

$$\gamma^{0n} = \underset{3}{\gamma^{0n}} + \underset{4}{\gamma^{0n}} + \ldots, \tag{2.5.29b}$$

$$\gamma^{mn} = \underset{4}{\gamma^{mn}} + \ldots . \tag{2.5.29c}$$

Because of (2.5.5), this induces in the h's the development

$$h^{00} = \underset{2}{h^{00}} + \underset{3}{h^{00}} + \underset{4}{h^{00}} + \ldots, \tag{2.5.30a}$$

$$h^{0n} = \underset{3}{h^{0n}} + \underset{4}{h^{0n}} + \ldots, \tag{2.5.30b}$$

$$h^{mn} = \underset{2}{h^{mn}} + \underset{3}{h^{mn}} + \underset{4}{h^{mn}} + \ldots \tag{2.5.30c}$$

where

$$\underset{2}{h^{00}} = -\underset{2}{h_{00}} = \tfrac{1}{2}\underset{2}{\gamma^{00}}, \quad \underset{2}{h^{mn}} = -\underset{2}{h_{mn}} = \tfrac{1}{2}\delta^{mn}\underset{2}{\gamma^{00}}, \quad \underset{3}{h^{0n}} = \underset{3}{h_{0n}} = \underset{3}{\gamma^{0n}}. \tag{2.5.31}$$

Therefore we know how to start our approximation procedure The next question is how to prolong it. That is: we assume

$$\underset{2}{\gamma^{00}}, \underset{3}{\gamma^{00}}, \ldots, \underset{l-1}{\gamma^{00}}; \underset{4}{\gamma^{mn}}, \underset{5}{\gamma^{mn}}, \ldots, \underset{l-1}{\gamma^{mn}}; \underset{3}{\gamma^{0n}}, \underset{4}{\gamma^{0n}}, \ldots, \underset{l}{\gamma^{0n}} \tag{2.5.32}$$

as known. The question is how to find $\underset{l}{\gamma^{mn}}, \underset{l}{\gamma^{00}}, \underset{l+1}{\gamma^{0n}}$. We shall look for the γ's in the order in which they have just been written out. Let us write the field equations of G. R. T. in the l'th order starting from the equation corresponding to (2.5.18c):

$$\underset{l}{K^{ma,nb}}{}_{|ab} + \underset{l-1}{K^{m0,nb}}{}_{\underset{1}{|0b}} + \underset{l-1}{K^{ma,n0}}{}_{\underset{1}{|a0}} + \underset{l-2}{K^{m0,n0}}{}_{\underset{2}{|00}} + N(\mathfrak{G}^{mn}) + 8\pi \underset{l}{\mathcal{T}^{mn}} = 0\,. \tag{2.5.33}$$

The $\underset{l}{\gamma^{mn}}$ are only present in the first expression. All others are known. Indeed $N(\mathfrak{G}^{mn})$ can contain the γ's mostly to the order $l-2$. And now, a few words about $\underset{l}{\mathcal{T}^{mn}}$. In the case of point particles we have according to (2.3.5):

$$\mathcal{T}^{mn} = \sum_{A=1}^{N} \overset{A}{\mu}\,\overset{A}{\delta}_{(3)}\,\overset{A}{\xi}{}^{m}{}_{|0}\,\overset{A}{\xi}{}^{n}{}_{|0}, \qquad \overset{A}{\mu} = \frac{k\overset{A}{m}_{(0)}}{c^2}\frac{dx^0}{ds_A}. \tag{2.5.34}$$

Now, we have:

$$\mu = \underset{2}{\mu}+\underset{3}{\mu}+\underset{4}{\mu}+\dots \tag{2.3.7}$$

However, if we start the h's according to (2.5.30), we see that $\underset{3}{\mu} = 0$. Therefore: in $\underset{l}{\mathcal{T}}^{mn}$ only expressions in the γ's to the order $l-4$ can appear. Similarly in $\underset{l}{\mathcal{T}}^{0n}$ only expressions to the order $l-3$ and finally in $\underset{l}{\mathcal{T}}^{00}$ only those to $l-2$ can appear. Furthermore, because $\underset{3}{\mu} = 0$, the development of $\mathcal{T}^{\alpha\beta}$, if we develop in it only the μ's, is:

$$\mathcal{T}^{00} = \underset{2}{\mathcal{T}}^{00}+\underset{4}{\mathcal{T}}^{00}+\dots, \tag{2.5.35a}$$

$$\mathcal{T}^{0n} = \underset{3}{\mathcal{T}}^{0n}+\underset{5}{\mathcal{T}}^{0n}+\dots, \tag{2.5.35b}$$

$$\mathcal{T}^{mn} = \underset{4}{\mathcal{T}}^{mn}+\underset{6}{\mathcal{T}}^{mn}+\dots. \tag{2.5.35c}$$

We assume that the same is true on the level of an arbitrary momentum tensor. That is: we assume that $\mathcal{T}^{\alpha\beta}$ depends on the γ's in such a way that only the known γ's appear in it. Besides, we assume that generally $\underset{3}{\mathcal{T}}^{00} = \underset{4}{\mathcal{T}}^{0n} = \underset{5}{\mathcal{T}}^{mn} = 0$. Therefore we can rewrite the differential equation for $\underset{l}{\gamma}^{mn}$ in the form:

$$\underset{l}{K}^{ma,nb}{}_{|ab} = \underset{l}{\Omega}^{mn}, \tag{2.5.36}$$

where $\underset{l}{\Omega}^{mn}$ is a known function of the known γ's. Because of the symmetry properties of $K^{ma,nb}$, we have

$$\underset{l}{K}^{ma,nb}{}_{|abn} \equiv 0. \tag{2.5.37}$$

Therefore

$$\underset{l}{\Omega}^{mn}{}_{|n} = 0 \tag{2.5.38}$$

is a necessary condition for equations (2.5.36) to be solvable. We shall discuss this condition in the next section. We shall see that it is connected with the equations of motion. However, at the moment, we shall simply assume that (2.5.38) is valid.

From (2.5.21) we know the general solution of the homogeneous equation (2.5.36). Denoting by $\underset{l}{\gamma}^{*mn}$ any particular solution of (2.5.36) we have the general solution:

$$\underset{l}{\gamma}^{mn} = \underset{l}{\gamma}^{*mn} + \underset{l}{b}^{m}{}_{|n} + \underset{l}{b}^{n}{}_{|m} - \delta^{mn}\underset{l}{b}^{s}{}_{|s}\,. \tag{2.5.39}$$

We may look at this equation in two different ways: either we regard the b's as arbitrary functions added to $\underset{l}{\gamma}^{*mn}$ so as to obtain the general solution, or we regard them as a transformation from the starred to the un-starred coordinate system. Indeed, a comparison of the last equation with (2.5.16) makes this point clear.

Since $\underset{l}{\gamma}^{*mn}$ is any particular solution, it is convenient to assume that

$$\underset{l}{\gamma}^{*mn}{}_{|n} = 0\,. \tag{2.5.40}$$

Indeed, if (2.5.40) is not fulfilled we can find a new solution

$$\underset{l}{\gamma}^{**mn} = \underset{l}{\gamma}^{*mn} + \underset{l}{b}^{*m}{}_{|n} + \underset{l}{b}^{*n}{}_{|m} - \delta^{mn}\underset{l}{b}^{*s}{}_{|s}\,, \tag{2.5.41}$$

so that

$$\underset{l}{\gamma}^{**mn}{}_{|n} = 0\,. \tag{2.5.42}$$

To do this, it is sufficient to solve the Poisson equation

$$\underset{l}{b}^{*m}{}_{|nn} = -\underset{l}{\gamma}^{*mn}{}_{|n}\,. \tag{2.5.43}$$

Thus, we can always assume that (2.5.40) is fulfilled. Then the differential equation (2.5.36) reduces to a Poisson equation

$$\underset{l}{\gamma}^{*mn}{}_{|ss} = 2\underset{l}{\Omega}^{mn}\,, \tag{2.5.44}$$

and we find the general solution by adding the solution of the homogeneous equation $\underset{l}{K}^{ma,nb}{}_{|ab} = 0$.

Having thus disposed of the γ^{mn} we turn to the γ^{00}'s. The proper equation for them, according to (2.5.18a), is

$$\underset{l}{K}^{0a,0b}{}_{|ab} = \underset{l}{\Omega}^{00} = -\mathcal{N}(\underset{l}{\mathfrak{G}}^{00}) - 8\pi\underset{l}{\mathcal{T}}^{00} \tag{2.5.45}$$

where $\underset{l}{\Omega}^{00}$ is a function of the already known γ's. According to (2.5.19c) we have

$$\underset{l}{K}^{0a,0b}{}_{|ab} = \tfrac{1}{2}\underset{l}{\gamma}^{00}{}_{|ss} - \tfrac{1}{2}\underset{l}{\gamma}^{ab}{}_{|ab}. \tag{2.5.46}$$

But the $\underset{l}{\gamma}^{ab}$ are already known from the previous step. We found them as the general solution of (2.5.36):

$$\underset{l}{\gamma}^{ab} = \underset{l}{\gamma}^{*ab} + \underset{l}{b}^{a}{}_{|b} + \underset{l}{b}^{b}{}_{|a} - \delta^{ab}\underset{l}{b}^{s}{}_{|s} \tag{2.5.47a}$$

$$\underset{l}{\gamma}^{*ab}{}_{|b} = 0, \qquad \underset{l}{\gamma}^{*ab}{}_{|ss} = 2\underset{l}{\Omega}^{ab}. \tag{2.5.47b}$$

Introducing this into (2.5.46), we have:

$$\tfrac{1}{2}\underset{l}{\gamma}^{00}{}_{|ss} = \tfrac{1}{2}\underset{l}{b}^{s}{}_{|srr} + \underset{l}{\Omega}^{00}. \tag{2.5.48}$$

Or, if we call $\underset{l}{\gamma}^{*00}$ the solution of

$$\underset{l}{\gamma}^{*00}{}_{|ss} = 2\underset{l}{\Omega}^{00}, \tag{2.5.49}$$

we find

$$\underset{l}{\gamma}^{00} = \underset{l}{\gamma}^{*00} + \underset{l}{b}^{s}{}_{|s}, \tag{2.5.50}$$

which, according to (2.5.16a) can again be interpreted as a change in a coordinate system.

Finally we have to consider the equation of the order $l+1$ for $\underset{l+1}{\gamma}^{0n}$. This is:

$$\underset{l+1}{K}^{0a,nb}{}_{|ab} + \underset{l+1}{K}^{0a,n0}{}_{|a0} + \underset{l+1}{N}(\mathfrak{G}^{0n}) + 8\pi\underset{l+1}{\mathcal{T}}^{0n} = 0. \tag{2.5.51}$$

Again, according to (2.5.19b), only $\underset{l+1}{K}^{0a,nb}$ contains the unknown quantities $\underset{l+1}{\gamma}^{0n}$. On the other hand, according to (2.5.19c), $\underset{l+1}{K}^{0a,n0}{}_{|a0}$ contains the quantities $\underset{l}{\gamma}^{mn}$ and $\underset{l}{\gamma}^{00}$ just calculated. We denote:

$$\underset{l+1}{\Omega}^{0n} = -\underset{l+1}{N}(\mathfrak{G}^{0n}) - 8\pi\underset{l+1}{\mathcal{T}}^{0n}. \tag{2.5.52}$$

Then, introducing the values for $\underset{l}{\gamma}^{mn}$ and $\underset{l}{\gamma}^{00}$ already calculated, that is (2.5.47) and (2.5.50), we have, because of (2.5.19) and (2.5.18b):

$$\tfrac{1}{2}(\underset{l+1}{\gamma}^{0n}+\underset{l+1}{b}^{n}{}_{|0})_{|ss}-\tfrac{1}{2}(\underset{l+1}{\gamma}^{0a}+\underset{l+1}{b}^{a}{}_{|0})_{|an}=\underset{l+1}{\Omega}^{0n}+\tfrac{1}{2}\underset{l}{\gamma}^{*00}{}_{\underset{1}{|0n}}. \tag{2.5.53}$$

Here the necessary condition for these equations to have a solution is the vanishing of the divergence of the right-hand side, which contains functions already known. That is:

$$\underset{l+1}{\Omega}^{0n}{}_{|n}+\underset{l+1}{\Omega}^{00}{}_{|0}=0. \tag{2.5.54}$$

We shall discuss this condition again later but for the moment we shall assume that it is fulfilled.

Let us now designate by $\underset{l+1}{\gamma}^{*0n}$ any particular solution of

$$\tfrac{1}{2}\underset{l+1}{\gamma}^{*0n}{}_{|ss}-\tfrac{1}{2}\underset{l+1}{\gamma}^{*0s}{}_{|sn}=\underset{l+1}{\Omega}^{0n}+\tfrac{1}{2}\underset{l}{\gamma}^{*00}{}_{\underset{1}{|0n}}. \tag{2.5.55}$$

Then

$$\underset{l+1}{\gamma}^{0n}+\underset{l+1}{b}^{n}{}_{|0}=\underset{l+1}{\gamma}^{*0n}+\underset{l+1}{b}^{0}{}_{|n}, \tag{2.5.56}$$

where $\underset{l+1}{b}^{0}$ is an arbitrary function, is again a general solution of (2.5.53). To the last equation we may easily add the condition:

$$\underset{l+1}{\gamma}^{*0a}{}_{|a}=\underset{l+1}{\gamma}^{*0n}{}_{|n}+\underset{l}{\gamma}^{*00}{}_{\underset{1}{|0}}=0. \tag{2.5.57}$$

If this condition is not fulfilled, we can find a different particular solution

$$\underset{l+1}{\gamma}^{**0n}=\underset{l+1}{\gamma}^{*0n}+\underset{l+1}{b}^{*0}{}_{|n} \tag{2.5.58}$$

fulfilling this condition. It is only necessary to choose $\underset{l+1}{b}^{*0}$ so that it is a solution of Poisson's equation:

$$\underset{l+1}{b}^{*0}{}_{|nn}=-\underset{l+1}{\gamma}^{*0n}{}_{|n}-\underset{l+1}{\gamma}^{*00}{}_{|0}.$$

For a $\gamma^{*0\alpha}$ fulfilling (2.5.57) the equation (2.5.56) again becomes a pure Poisson equation:

$$\underset{l+1}{\gamma}^{*0m}{}_{|ss} = 2 \underset{[l+1}{\Omega}^{0m}. \tag{2.5.59}$$

As before, we may interpret (2.5.56) as a coordinate transformation.

Let us summarize briefly what has been done in this section. Firstly, we have shown that by choosing a simple coordinate system we can start our development with $\underset{2}{\gamma}^{00}, \underset{3}{\gamma}^{0n}, \underset{4}{\gamma}^{mn}$. Then, having started it, we can prolong it indefinitely, assuming that the integrability conditions, as expressed by (2.5.38) and (2.5.54), are fulfilled. We can find particular solutions step by step by solving only Poisson's equations. Then we can find the general solutions by adding appropriate arbitrary functions. Their addition may be interpreted as passing from an especially chosen to an arbitrary coordinate system. Of course, we do not know whether such an approximation procedure converges; nor do we know whether it behaves decently at infinity.

We have shown that we can always find a coordinate system such that the γ's start according to (2.5.29). But we can show more; we can show that by a choice of an appropriate coordinate system we have:

$$\gamma^{00} = \underset{2}{\gamma}^{00} + \underset{4}{\gamma}^{00} + \underset{5}{\gamma}^{00} + \dots, \tag{2.5.60a}$$

$$\gamma^{0n} = \underset{3}{\gamma}^{0n} + \underset{5}{\gamma}^{0n} + \underset{6}{\gamma}^{0n} + \dots, \tag{2.5.60b}$$

$$\gamma^{mn} = \underset{4}{\gamma}^{mn} + \underset{6}{\gamma}^{mn} + \underset{7}{\gamma}^{mn} + \dots. \tag{2.5.60c}$$

This can easily be shown on the level of field singularities. Indeed because $\underset{3}{\mu} = 0$, the corresponding $\underset{3}{\mathcal{T}}^{00}, \underset{4}{\mathcal{T}}^{0n}, \underset{5}{\mathcal{T}}^{mn}$ vanish. Moreover $N(\underset{3}{\mathfrak{G}}^{00})$ vanishes, because it is at least of the fourth order. $N(\underset{4}{\mathfrak{G}}^{0n})$ also vanishes since the only contribution to it would have to come from expressions like $\underset{3}{\gamma}^{00}{}_{|0}\underset{2}{\gamma}^{00}{}_{|n}$ which are of the fifth order.

$\underset{5}{N}(\mathfrak{G}^{mn})$ also vanishes since the only contribution to it would have to come from expressions of the type $\underset{4}{\gamma}^{0m}{}_{|0}\underset{2}{\gamma}^{00}{}_{|n}$ which are of at least the sixth order. If we wish to obtain the post-Newtonian equations of motion, then for practical reasons only the disappearance of $\underset{3}{\gamma}^{00}$ is of some interest.

6. ON THE TWO FORMS OF THE EQUATIONS OF MOTION AND THE INTEGRABILITY CONDITIONS

We shall now return to the general theory. We remember that, again on the level of field singularities, the equations of motion for the A'th particle were

$$\frac{1}{c^2}\frac{dx^0}{ds_A}\int\limits_{\overset{A}{\Omega}}\mathfrak{T}^{\alpha\beta}{}_{;\beta}\,d\mathbf{x} = 0 \tag{2.6.1}$$

where $\overset{A}{\Omega}$ was a small neighbourhood surrounding the A'th singularity. This form is derived immediately from the symbolic equation

$$\mathfrak{T}^{\alpha\beta}{}_{;\beta} = 0 \tag{2.6.2}$$

which is a consequence of both the field equations and the Bianchi identities. We shall regard the last equation as that of motion on the level of continuous distributions and the arbitrary energy-momentum tensor.

We shall now introduce a different form of the equations of motion historically prior to this one; in fact it was from this form (which we shall introduce now and return to in the last chapter) that all the development of the equations of motion in G. R. T. originated.

We write the field equations of G. R. T., according to (2.5.18), in the form:

$$\begin{aligned}\mathfrak{G}^{\mu n}+8\pi\mathfrak{T}^{\mu n} = K^{\mu a,nb}{}_{|ab}+K^{\mu 0,nb}{}_{|0b}+K^{\mu a,n0}{}_{|a0}+\\ +K^{\mu 0,n0}{}_{|00}+N(\mathfrak{G}^{\mu n})+8\pi\mathfrak{T}^{\mu n} = 0.\end{aligned} \tag{2.6.3}$$

Now let us differentiate the last equation with respect to x^n. Because of the structure of the first two expressions on the right-hand side, their derivatives disappear and what remains is:

$$8\pi \mathcal{T}^{\mu n}{}_{|n} + K^{\mu a,n0}{}_{|an0} + K^{\mu 0,n0}{}_{|n00} + N(\mathfrak{G}^{\mu n})_{|n} = 0. \qquad (2.6.4)$$

These are four equations gained from the field equations which we assumed to be satisfied; they owe their form to the structure of the field equations.

Also because of the structure of the field equations as expressed by Bianchi identities, if the field equations are fulfilled, we obtain the equations of motion:

$$\mathfrak{T}^{\mu n}{}_{|n} + \mathfrak{T}^{\mu 0}{}_{|0} + \begin{Bmatrix} \mu \\ \alpha\beta \end{Bmatrix} \mathfrak{T}^{\alpha\beta} = 0. \qquad (2.6.5)$$

The two equations are identical in their physical contents. Both form conditions that must be satisfied; both are different expressions of the equations of motion.

Let us split (2.6.4), writing out separately the equations for $\mu = m$ and for $\mu = 0$:

$$8\pi \mathcal{T}^{mn}{}_{|n} + K^{ma,n0}{}_{|an0} + K^{m0,na}{}_{|an0} + K^{m0,n0}{}_{|n00} + N(\mathfrak{G}^{mn})_{|n} = 0 \quad (2.6.6a)$$

$$8\pi \mathcal{T}^{0n}{}_{|n} + K^{0a,n0}{}_{|an0} + N(\mathfrak{G}^{0n})_{|n} = 0. \qquad (2.6.6b)$$

If l had been written under the first equation and $l+1$ unde- the second, these would have been precisely the integrability condir tions mentioned in the last section in (2.5.38) and (2.5.54). Thus there is a very intimate connection between the equations of motion and the integrability condition. But there is one question which remains to be answered: how can we cut equations (2.6.6) or equations (2.6.5) into pieces corresponding to different approximations? This question was the essential stumbling block in the work of Einstein, Infeld and Hoffmann. It seemed that cutting up the equations of motion would give an infinite number of contradictory equations. To answer the question, or rather to find one of its possible answers, we must deepen our approximation procedure.

Since we are more familiar with the form (2.6.5) than (2.6.6)

it will be better to explain this procedure for the equation $\mathfrak{T}^{\alpha\beta}{}_{;\beta} = 0$. Also, to make the ideas as simple as possible, we shall remain on the level of field singularities. We know, then, that by integrating over a small volume surrounding the A'th singularity we can obtain equations in the form:

$$\frac{d}{dx^0}(\overset{A}{\mu}\overset{A}{\xi}{}^k{}_{|0}) + \overset{A}{\mu}\overset{A}{\overline{\begin{Bmatrix} k \\ \alpha\beta \end{Bmatrix}}}\overset{A}{\xi}{}^\alpha{}_{|0}\overset{A}{\xi}{}^\beta{}_{|0} = 0, \tag{2.6.7a}$$

$$\frac{d}{dx^0}\overset{A}{\mu} + \overset{A}{\mu}\overset{A}{\overline{\begin{Bmatrix} 0 \\ \alpha\beta \end{Bmatrix}}}\overset{A}{\xi}{}^\alpha{}_{|0}\overset{A}{\xi}{}^\beta{}_{|0} = 0. \tag{2.6.7b}$$

Now let us isolate from (2.6.7a) all expressions of the l'th order. These are in $\dfrac{d}{dx^0}(\overset{A}{\mu}\overset{A}{\xi}{}^k{}_{|0})$:

$$\underset{3}{\overset{A}{\mu}}{}_{|0}\underset{l-3}{\overset{A}{\xi}}{}^k{}_{|0} + \underset{2}{\overset{A}{\mu}}\underset{l-2}{\overset{A}{\xi}}{}^k{}_{|00} + \underset{5}{\overset{A}{\mu}}{}_{|0}\underset{l-5}{\overset{A}{\xi}}{}^k{}_{|0} + \underset{4}{\overset{A}{\mu}}\underset{l-4}{\overset{A}{\xi}}{}^k{}_{|00} + \ldots + \underset{l-1}{\overset{A}{\mu}}\underset{1}{\overset{A}{\xi}}{}^k{}_{|0}. \tag{2.6.8}$$

In the second expression we shall have in the lowest order:

$$\underset{2}{\overset{A}{\mu}}\underset{2}{\overline{\gamma^{00}}}{}_{|k} \quad \text{with} \quad \underset{2}{\overline{\gamma^{00}}}(x^\alpha, \overset{A}{\xi}{}^\alpha).$$

Because $\underset{2}{\gamma^{00}}$ depends on $\overset{A}{\xi}{}^\alpha$, this will also contain contributions up to the order l. They will come from the development of $\overset{A}{\xi}{}^\alpha$ inside $\underset{2}{\gamma^{00}}$. Because $\underset{2}{\mu}\underset{2}{\gamma^{00}}{}_{|k}$ is of the order 4, we shall have to develop $\overset{A}{\xi}{}^\alpha$ up to the order $l-4$. In other expressions (in 2.6.7a) we shall have to develop $\overset{A}{\xi}{}^\alpha$ up to an order lower than $l-4$.

Therefore, the highest order expression appearing in $\overset{A}{\xi}{}^k$ is of the order $l-4$. Therefore, we can now regard the equation

$$\underbrace{\frac{d}{dx^0}(\overset{A}{\mu}\overset{A}{\xi}{}^k{}_{|0})}_{l} + \underbrace{\overset{A}{\mu}\overset{A}{\overline{\begin{Bmatrix} k \\ \alpha\beta \end{Bmatrix}}}}_{l}\overset{A}{\xi}{}^\alpha{}_{|0}\overset{A}{\xi}{}^\beta{}_{|0} = 0 \tag{2.6.9}$$

as defining $\overset{A}{\underset{l-4}{\xi}}{}^{k}$. This means: the equations of motion can be cut into different orders, each of which allows us to calculate motion with an increasing accuracy. The equations of motion of order l allow us to calculate motion up to the order $l-4$. Thus, if we put $l = 4$, we have the equation for $\underset{0}{\xi}{}^{k}$, that is the Newtonian motion. Putting $l = 6$ we have $\underset{2}{\xi}{}^{k}$, the post-Newtonian motion.

Now, what about the equation of the order $l+1$:

$$\underset{l+1}{\mu}{}_{|0} + \underbrace{\mu \begin{Bmatrix} 0 \\ \alpha\beta \end{Bmatrix} \xi^{\alpha}{}_{|0}\, \xi^{\beta}{}_{|0}}_{l+1} = 0? \tag{2.6.10}$$

This equation simply gives us $\underset{l}{\mu}$! If we take the order $l+1 = 3$ we obtain simply $\underset{2}{\mu} = \text{constant}$ and for $l+1 = 5$ we obtain the first additional, non-Newtonian, expression.

Thus we see that we may cut the equations of motion and obtain the motion with an increasing accuracy. We must assume the same situation to be true on the level of the general energy-momentum tensor. There is the dynamical characteristic A^{a} which plays the role that ξ^{a} does on the lower level. It may also be found by the approximation procedure with an increasing accuracy.

All that has been said with respect to the equations of motion, applies just as well to our integrability conditions. They determine $\underset{l-4}{\xi}{}^{k}$ and $\underset{l}{\mu}$. This can be seen from (2.6.6a) if we write l underneath:

$$8\pi \underset{l}{\mathcal{T}}{}^{mn}{}_{|n} + \underset{l}{K}{}^{ma,n0}{}_{|an0} + K^{m0,na}{}_{|an0} + \underset{l}{K}{}^{m0,n0}{}_{|n00} + N(\underset{l}{\mathfrak{G}}{}^{mn})_{|n} = 0 \tag{2.6.11}$$

In $\underset{l}{\mathcal{T}}{}^{mn}$ the highest order of ξ^{k} is $l-4$:

$$\underset{l}{\mathcal{T}}{}^{mn} = \sum_{A=1}^{N} \overset{A}{\underset{2}{\mu}}\, \overset{A}{\underset{0}{\xi}}{}^{m}{}_{|0}\, \overset{A}{\underset{l-4}{\xi}}{}^{n}{}_{|0}\, \overset{A}{\underset{1}{\delta}} + \ldots \tag{2.6.12}$$

Similarly in the $\underset{4}{\gamma}$'s appearing in the K's and $\mathfrak{G}$'s, ξ^{m} must be devel-

oped up to the order $l-4$. Thus we have exactly the same situation as before.

Summarizing: the integrability conditions are equations of motion cut like the field equations. They allow us to find the motion with an increasing accuracy.

The procedure outlined here is almost unnecessary if we do not go beyond the post-Newtonian motion. In general it is somewhat "messy" since it mixes up the development of the γ's with that of the ξ's or, in the general case, with the A's.

In the original paper of Einstein, Infeld and Hoffmann a different procedure was used. The authors of this book later perfected this procedure and called it the dipole procedure. From the practical point of view it is of little use, but the theory built around it is more elegant than that first given, because it avoids the development of the motion itself. We shall sketch the use of the dipole theory only briefly in words rather than in detailed calculations.

The dipole method was outlined in the last section of the first chapter. We recall that the idea of this method consisted in the introduction of an artificial vector field D^a. It was introduced in such a way that the Bianchi identities were identically fulfilled. This means that the equations of motion were identically fulfilled for arbitrary motion. In this way, by introducing the artificial field D^a, by abandoning Einstein's equations, we tore apart the connection between field and motion. We could find the field for these non-Einsteinian equations and the motion remained arbitrary. Thus nothing stood in the way of cutting the equations and finding the field and the D's step by step. Thus we could find step by step the $\underset{l}{\gamma}^{mn}, \underset{l}{\gamma}^{00}, \underset{l+1}{\gamma}^{0n}, \underset{l}{D}^{k}, \underset{l+1}{D}^{0}$. Let us assume that we stop here. Then we can change the solution of the non-Einsteinian equations into that of the Einsteinian equations by putting

$$\underset{4}{D}^k+\underset{5}{D}^k+\ldots+\underset{l}{D}^k=0, \tag{2.6.13a}$$

$$\underset{3}{D}^0+\underset{4}{D}^0+\ldots+\underset{l+1}{D}^0=0. \tag{2.6.13b}$$

This means that only at the end of our approximation procedure do we find the equations of motion, which had been arbitrary until then. Indeed, from the last equations it follows, according to (1.10.11) that:

$$\underset{4}{\mathcal{T}}^{k\mu}{}_{;\mu}+\underset{5}{\mathcal{T}}^{k\mu}{}_{;\mu}+\ldots+\underset{l}{\mathcal{T}}^{k\mu}{}_{;\mu} = 0, \tag{2.6.14a}$$

$$\underset{3}{\mathcal{T}}^{0\mu}{}_{;\mu}+\underset{4}{\mathcal{T}}^{0\mu}{}_{;\mu}+\ldots+\underset{l+1}{\mathcal{T}}^{0\mu}{}_{;\mu} = 0. \tag{2.6.14b}$$

Thus by this method we obtain the equations of motion at once when our calculations of the field are finished.

In principle, therefore, two methods of procedure are possible. The first consists in developing the masses and the motion in series:

$$\begin{aligned} \mu &= \underset{2}{\mu}+\underset{4}{\mu}+\ldots, \\ \xi^k &= \underset{0}{\xi}^k+\underset{2}{\xi}^k+\ldots. \end{aligned} \tag{2.6.15}$$

By cutting the equations of motion into pieces, we can find consecutively .

$$\underset{2}{\mu},\ \underset{4}{\mu},\ \underset{5}{\mu},\ \ldots; \qquad \underset{0}{\xi}^k,\ \underset{2}{\xi}^k,\ \underset{3}{\xi}^k,\ \ldots. \tag{2.6.16}$$

We shall say that such methods consist of the use of the approximation procedure of a certain order.

In the other method, as explained for the dipole procedure, we do not split the equations of motion. We regard

$$\mu,\ \xi^k \tag{2.6.17}$$

as arbitrary, which, at the end of our calculations, we find at once up to a certain order, that is, we find:

$$\underset{\to n+2}{\mu},\ \underset{\to n}{\xi}^k. \tag{2.6.18}$$

The arrows denote that the μ's and ξ's are not of a certain order but up to a certain order.

As far as our calculations of the Newtonian and post-Newtonian motion are concerned, there is little difference between these two

methods. We shall proceed in the following way: we shall start with the Newtonian equations of motion:

$$\underset{3}{\mathcal{T}}^{0\nu}{}_{;\nu} = 0; \qquad \underset{4}{\mathcal{T}}^{m\nu}{}_{;\nu} = 0. \tag{2.6.19}$$

Then we shall procced to the post-Newtonian equations of motion:

$$\begin{aligned} \underset{3}{\mathcal{T}}^{0\nu}{}_{;\nu} + \underset{5}{\mathcal{T}}^{0\nu}{}_{;\nu} &= 0, \\ \underset{4}{\mathcal{T}}^{m\nu}{}_{;\nu} + \underset{6}{\mathcal{T}}^{m\nu}{}_{;\nu} &= 0. \end{aligned} \tag{2.6.20}$$

However, into the $\underset{5}{\mathcal{T}}$ and $\underset{6}{\mathcal{T}}$ we shall introduce the Newtonian solution already known from the previous approximation steps.

These remarks finish our general theory of the approximation method and we may now pass on to actual calculations.

CHAPTER III

THE NEWTONIAN AND POST-NEWTONIAN APPROXIMATION

1. THE NEWTONIAN APPROXIMATION

In this chapter we shall find the actual equations of motion, using the approximation method already described. We shall start, of course, with the Newtonian approximation, and shall find both the Newtonian and post-Newtonian approximations on the three levels outlined in the previous chapter, starting with the level of the general energy-momentum tensor and then descending to the level of continuous and finally, discontinuous mass distribution. Thus the Newtonian equations of motion, that is those of the lowest order, are:

$$\underset{2}{\mathcal{T}}^{00}{}_{\underset{1}{|0}}+\underset{3}{\mathcal{T}}^{0a}{}_{|a}=0, \tag{3.1.1a}$$

$$\underset{3}{\mathcal{T}}^{0m}{}_{\underset{1}{|0}}+\underset{4}{\mathcal{T}}^{mn}{}_{|n}+\tfrac{1}{2}\underset{2}{h}_{00|m}\underset{2}{\mathcal{T}}^{00}=0. \tag{3.1.1b}$$

Therefore, all that must be known in order to write out these equations of motion is $\underset{2}{h}_{00}$. But, according to (2.5.31) and (2.5.27):

$$\underset{2}{h}_{00}=-\tfrac{1}{2}\underset{2}{\gamma}^{00}, \qquad \underset{2}{\gamma}^{00}{}_{|ss}=-16\pi\underset{2}{\mathcal{T}}^{00}. \tag{3.1.2}$$

If we assume that $\underset{2}{h}_{00}$ vanishes at infinity it follows that:

$$\underset{2}{h}_{00}=-2\int_{\Omega}d\mathbf{x}'\underset{2}{\mathcal{T}}'^{00}|\mathbf{x}-\mathbf{x}'|^{-1}\,. \tag{3.1.3}$$

From the $'$ in $\underset{2}{\mathcal{T}}'^{00}$ we understand that the arguments are x'^k and x^0. The arbitrary additive harmonic function vanishes and Ω denotes the entire space. Therefore, the Newtonian equations of motion are:

$$\underset{2}{\mathcal{T}}^{00}{}_{|0} + \underset{3}{\mathcal{T}}^{0\alpha}{}_{|\alpha} = 0, \tag{3.1.4a}$$

$$\underset{3}{\mathcal{T}}^{0m}{}_{|0} + \underset{1}{}\underset{4}{\mathcal{T}}^{mn}{}_{|n} - \int_{\Omega} d\mathbf{x}' \underset{2}{\mathcal{T}}'^{00} \underset{2}{\mathcal{T}}^{00} \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)_{|m} = 0. \tag{3.1.4b}$$

Now we descend to the case of a perfect fluid characterized by the energy-momentum tensor:

$$\mathcal{T}^{\alpha\beta} = \frac{k}{c^2}\sqrt{-g}\left[\left(\varrho + \frac{\varrho}{c^2}\int_0^p \frac{dp}{\varrho}\right) u^\alpha u^\beta - \frac{p}{c^2} g^{\alpha\beta}\right] \tag{3.1.5}$$

and we assume that

$$p = p(\varrho) \qquad \text{or} \qquad \varrho = \varrho(p) \tag{3.1.6}$$

is given.

Following our general scheme, we are faced with the problem of establishing the order of the first expression in the development with respect to λ. Since u^k starts with the order one, u^0 with the order zero, $\sqrt{-g}$ with the order zero, then it follows from the form (3.1.5) that ϱ starts with the order two (as μ did before) and p with the order four. Thus we have:

$$\underset{2}{\mathcal{T}}^{00} = \underset{2}{\sigma}, \qquad \underset{3}{\mathcal{T}}^{0n} = \underset{2}{\sigma}\underset{1}{v}^n, \qquad \underset{4}{\mathcal{T}}^{mn} = \underset{2}{\sigma}\underset{1}{v}^m \underset{1}{v}^n + \underset{4}{\pi}\delta^{mn}, \tag{3.1.7}$$

where:

$$\underset{2}{\sigma} = \frac{k}{c^2}\sqrt{-g}\,\varrho, \quad v^\alpha = \frac{dx^\alpha}{dx^0}, \quad \underset{4}{\pi} = \frac{kp}{c^4}, \quad \underset{2}{h}_{00} = -2\int_{\Omega} d\mathbf{x}'\,\underset{2}{\sigma}'\,|\mathbf{x}-\mathbf{x}'|^{-1}.$$

Therefore the Newtonian equations of motion are:

$$\underset{2}{\sigma}_{|0} + (\underset{2}{\sigma}\underset{1}{v}^n)_{|n} = 0, \tag{3.1.8a}$$

$$(\underset{2}{\sigma}\underset{1}{v}^m)_{|0} + (\underset{2}{\sigma}\underset{1}{v}^m \underset{1}{v}^n)_{|n} + \underset{4}{\pi}_{|m} - \int_{\Omega} d\mathbf{x}'\,\underset{2}{\sigma}'\,\underset{2}{\sigma}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)_{|m} = 0. \tag{3.1.8b}$$

When multiplied by c^3 the first of these equations becomes precisely the continuity equation; if multiplied by c^4, the second becomes precisely the Newtonian equation for a perfect fluid.

Now let us descend to the case of point particles. We can do this in two ways: either substituting directly into the equations of motion

$$\mathcal{T}^{\alpha\beta} = \sum_{A} \overset{A}{\mu}\,\overset{A}{\delta}\,\overset{A}{\xi}{}^{\alpha}{}_{|0}\,\overset{A}{\xi}{}^{\beta}{}_{|0}, \tag{3.1.9}$$

or, as a limiting case of (3.1.8), by putting

$$\sigma = \sum_{A} \overset{A}{\mu}\,\delta(\mathbf{x} - \overset{A}{\boldsymbol{\xi}}). \tag{3.1.10}$$

We shall choose the second way, though it is slightly more complicated. Then by integrating equation (3.1.8a) over $\overset{A}{\Omega}$, and because the integral of the divergence can be changed into a surface integral with integrand vanishing at the surface we have:

$$\underset{2}{\mu}{}_{|0} = 0, \qquad \underset{2}{\mu} = \text{constant}. \tag{3.1.11}$$

Then for the same reason, the second integral gives:

$$\overset{A}{\mu}\,\overset{A}{\xi}{}^{m}{}_{|00} = \int_{\overset{A}{\Omega}} d\mathbf{x} \int_{\Omega} d\mathbf{x}' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right)_{|m} \sum_{A} \overset{A}{\mu}\,\delta(\mathbf{x}' - \overset{A}{\boldsymbol{\xi}}) \sum_{B} \overset{B}{\mu}\,\delta(\mathbf{x} - \overset{B}{\boldsymbol{\xi}})$$

$$= \sum_{B \neq A} \overset{A}{\mu}\overset{B}{\mu} \frac{\partial}{\partial \overset{A}{\xi}{}^{m}} \frac{1}{\left|\overset{A}{\boldsymbol{\xi}} - \overset{B}{\boldsymbol{\xi}}\right|} = \sum_{B}{}' \overset{A}{\mu}\,\overset{B}{\mu} \left(\frac{1}{\left|\overset{A}{\boldsymbol{\xi}} - \overset{B}{\boldsymbol{\xi}}\right|}\right)_{|\overset{A}{\xi}{}^{m}} \tag{3.1.12}$$

which are again the Newtonian equations of motion of point particles under the influence of the Newtonian gravitational field. Here the $'$ on Σ reminds us that the sum is to be extended over all B's different from A. This result does not depend on whether or not we used our good δ's, since the contribution from the term $B = A$ is always zero. This we see by interchanging $\mathbf{x}$ and $\mathbf{x}'$ for $A = B$ in (3.1.12).

2. THE GRAVITATIONAL FIELD FOR THE POST-NEWTONIAN EQUATIONS OF MOTION

To find the equations of motion in the next, or post-Newtonian approximation, we shall have to know:

$$\underset{4}{\mathcal{T}}^{00},\ \underset{5}{\mathcal{T}}^{0n},\ \underset{6}{\mathcal{T}}^{mn} \tag{3.2.1}$$

since, as was assumed in (2.5.35),

$$\underset{3}{\mathcal{T}}^{00} = \underset{4}{\mathcal{T}}^{0n} = \underset{5}{\mathcal{T}}^{mn} = 0. \tag{3.2.2}$$

As far as the gravitational field is concerned, we shall have to develop

$$\begin{Bmatrix} 0 \\ \mu\nu \end{Bmatrix} \mathcal{T}^{\mu\nu} \quad \text{and} \quad \begin{Bmatrix} \mathrm{m} \\ \mu\nu \end{Bmatrix} \mathcal{T}^{\mu\nu} \tag{3.2.3}$$

up to the fifth and sixth orders. To do this we must know:

$$\underset{2}{h}_{mn},\ \underset{3}{h}_{0n},\ \underset{4}{h}_{00}. \tag{3.2.4}$$

Therefore we shall calculate them in turn.

We know from (2.5.31) that

$$\underset{2}{h}_{mn} = -\tfrac{1}{2}\,\delta^{mn}\underset{2}{\gamma}^{00} = \delta^{mn}\underset{2}{h}_{00} = -2\,\delta^{mn}\int d\mathbf{x}'\,\underset{2}{\mathcal{T}}'^{00}\,|\mathbf{x}-\mathbf{x}'|^{-1}. \tag{3.2.5}$$

For $\underset{3}{h}^{0n}$ we have, because of

$$\underset{3}{h}^{0n} = \underset{3}{\gamma}^{0n} = \underset{3}{h}_{0n}, \tag{3.2.6}$$

and because of (2.5.14) and (2.5.28):

$$\tfrac{1}{2}\underset{3}{h}_{0n|ss} - \tfrac{1}{2}\underset{3}{h}_{0s|sn} = -8\pi\underset{3}{\mathcal{T}}^{0n} + \tfrac{1}{2}\underset{2}{\gamma}^{00}{}_{|n\underset{1}{0}}. \tag{3.2.7}$$

The condition of integrability of this equation is, as can be seen from (3.1.1a),

$$\underset{2}{\mathcal{T}}^{00}{}_{|\underset{1}{0}} + \underset{3}{\mathcal{T}}^{0s}{}_{|s} = 0, \tag{3.2.8}$$

which is the zero equation of the Newtonian equations of motion. Assuming now that

$$\gamma^{*0\alpha}{}_{|\alpha} = 0, \tag{3.2.9}$$

we have from (3.2.7):

$$\tfrac{1}{2}\underset{3}{\overset{*}{\gamma}}_{0n|ss} = -8\pi\underset{3}{\mathcal{T}}^{0n}. \tag{3.2.10}$$

The general solution is, by Chapter II:

$$\underset{3}{\gamma}_{0n} = 4\int\limits_{\Omega} d\mathbf{x}'\,\frac{\underset{3}{\mathcal{T}}'^{0n}}{|\mathbf{x}-\mathbf{x}'|} + \underset{3}{a}_{0|n}, \tag{3.2.11}$$

where a_0 is an arbitrary function, if necessary fulfilling certain conditions of continuity and vanishing at infinity.

It is much more complicated to find $\underset{4}{h}_{00}$. The simplest way is not through the γ's, that is not through the Einstein tensor $\mathfrak{G}^{\mu\nu}$, but through the Ricci tensor density $\mathfrak{R}^{00}$. Let us write once more the principal equations of G. R. T.

$$\mathfrak{R}^{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta}\mathfrak{R} = -8\pi\mathcal{T}^{\alpha\beta}. \tag{3.2.12}$$

From these we find

$$\mathfrak{R} = g^{\alpha\beta}\mathfrak{R}_{\alpha\beta} = 8\pi\mathcal{T}, \quad \mathcal{T} = g_{\alpha\beta}\mathcal{T}^{\alpha\beta}. \tag{3.2.13}$$

Therefore, instead of (3.2.12) we may write:

$$\mathfrak{R}^{\alpha\beta} = -8\pi(\mathcal{T}^{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta}\mathcal{T}). \tag{3.2.14}$$

Let us once more write $R_{\mu\nu}$ explicitly:

$$R_{\mu\nu} = \begin{Bmatrix}\varrho\\ \mu\varrho\end{Bmatrix}_{|\nu} - \begin{Bmatrix}\varrho\\ \mu\nu\end{Bmatrix}_{|\varrho} + \begin{Bmatrix}\varrho\\ \mu\sigma\end{Bmatrix}\begin{Bmatrix}\sigma\\ \varrho\nu\end{Bmatrix} - \begin{Bmatrix}\varrho\\ \mu\nu\end{Bmatrix}\begin{Bmatrix}\sigma\\ \varrho\sigma\end{Bmatrix}. \tag{3.2.15}$$

For brevity we write:

$$\underset{2}{h}_{00} = \varphi = -2\int\limits_{\Omega} d\mathbf{x}'\,\underset{2}{\mathcal{T}}'^{00}\,|\mathbf{x}-\mathbf{x}'|^{-1}. \tag{3.2.16}$$

Therefore up to the order two we have, because of (3.2.5):

$$\sqrt{-g} = 1+\tfrac{1}{2}(h_{00}-h_{ss}) = 1-\varphi. \tag{3.2.17}$$

Since

$$\mathfrak{R}^{00} = \mathfrak{R}_{\mu\nu}g^{\mu 0}g^{\nu 0}, \tag{3.2.18}$$

we have, for $\mathfrak{R}^{00}$ of the order four:

$$\underset{4}{\mathfrak{R}}^{00} = -3\varphi \underset{2}{R}_{00}+\underset{4}{R}_{00}. \tag{3.2.19}$$

Thus the problem of finding $\underset{4}{\mathfrak{R}}^{00}$ is reduced to that of finding R_{00} up to the order four. A straightforward calculation gives:

$$\underset{2}{R}_{00}+\underset{4}{R}_{00} = -\tfrac{1}{2}\varphi_{|ss}-\tfrac{1}{2}\underset{4}{h}_{00|ss}+\underset{3}{h}_{0s|\underset{1}{0}s}-\tfrac{3}{2}\varphi_{|00}+\tfrac{1}{2}\varphi_{|s}\varphi_{|s}-\tfrac{1}{2}\varphi\varphi_{|ss}. \tag{3.2.20}$$

Because, according to (3.2.11) and (3.1.1a):

$$\underset{4}{h}_{0s|0s} = 2\varphi_{|00}+a_{0|0ss} \tag{3.2.21}$$

we can write $\underset{4}{\mathfrak{R}}^{00}$ in the form:

$$\underset{4}{\mathfrak{R}}^{00} = -\tfrac{1}{2}(\underset{4}{h}_{00}-2a_{0|0}-\tfrac{1}{2}\varphi^2)_{|ss}+\tfrac{1}{2}\varphi_{|00}+\tfrac{1}{2}\varphi\varphi_{|ss} \tag{3.2.22}$$

Now, on the right-hand side of (3.2.14) we have

$$\underset{4}{\mathfrak{R}}^{00} = -8\pi(\underset{4}{\mathcal{T}}^{00}-\underbrace{\tfrac{1}{2}g^{00}\mathcal{T}}_{4}) = -4\pi(\underset{4}{\mathcal{T}}^{00}+\underset{4}{\mathcal{T}}^{ss}). \tag{3.2.23}$$

Therefore our differential equation becomes:

$$(\underset{4}{h}_{00}-2a_{0|0}-\tfrac{1}{2}\varphi^2)_{|ss} = \varphi_{|00}+\varphi\varphi_{|ss}+8\pi(\underset{4}{\mathcal{T}}^{00}+\underset{4}{\mathcal{T}}^{ss}). \tag{3.2.24}$$

Because of the form of φ, (3.2.16), we obtain the following solution of the last equation:

$$\begin{aligned}\underset{4}{h}_{00} = {} & -2\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|^{-1}(\underset{4}{\mathcal{T}}'^{00}+\underset{4}{\mathcal{T}}'^{ss})-\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|\underset{2}{\mathcal{T}}'^{00}{}_{\underset{2}{|00}}+ \\ & +2\int d\mathbf{x}'\int d\mathbf{x}''\,[\mathbf{x},\mathbf{x}',\mathbf{x}'']\underset{2}{\mathcal{T}}'^{00}\underset{2}{\mathcal{T}}''^{00}+2\underset{4}{a}_{0|0},\end{aligned} \tag{3.2.25}$$

where

$$[\mathbf{x}, \mathbf{x}', \mathbf{x}''] = \frac{1}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|} + \frac{1}{|\mathbf{x}'-\mathbf{x}''||\mathbf{x}''-\mathbf{x}|} + \frac{1}{|\mathbf{x}''-\mathbf{x}||\mathbf{x}-\mathbf{x}'|}.$$

This formula reveals the structure of $\underset{4}{h}_{00}$ and all the relativistic influences upon it. It consists of four expressions. The first one is the usual contribution coming from the distribution of matter in the non-Newtonian approximation. The second would also have appeared in a linear theory. It results from the fact that we began with Laplace instead of d'Alembert equations. The third expression is due to the non-linearity of the field equations in G. R. T. The meaning of the fourth expression is obvious.

We should now like to find out how $\underset{4}{h}_{00}$ behaves at (spatial) infinity, (if $a_{0|0}$ goes to zero there). If we assume that all matter is contained in a sphere with a finite radius, then all the expressions, with the exception of the second, obviously go to zero. Thus it is necessary to examine more closely only the expression

$$-\int d\mathbf{x}' \, |\mathbf{x}-\mathbf{x}'| \underset{2}{\mathcal{T}}'^{00}{}_{|00}. \tag{3.2.26}$$

But we remember the integrability condition for $\underset{3}{\gamma}^{0n}$ which we have assumed is fulfilled:

$$-\underset{2}{\mathcal{T}}^{00}{}_{\underset{1}{|0}} = \underset{3}{\mathcal{T}}^{0s}{}_{|s}. \tag{3.2.27}$$

Therefore we may write (3.2.26):

$$|\mathbf{x}| \int d\mathbf{x}' \mathcal{T}'^{0s}{}_{|s0} - \frac{x^k}{|\mathbf{x}|} \int x'^k \mathcal{T}'^{0s}{}_{|s0} \, d\mathbf{x}' +$$

$$+ \frac{1}{2|\mathbf{x}|} \int d\mathbf{x}' x'^k x'^k \mathcal{T}'^{0s}{}_{|s0} - \frac{1}{2} \frac{x^k x^l}{|\mathbf{x}|^3} \int d\mathbf{x}' x'^k x'^l \mathcal{T}'^{0s}{}_{|0s} + \dots \tag{3.2.28}$$

The first expression is zero, because $\mathcal{T}'^{0s}$ vanishes beyond a finite sphere; the third and the fourth go to zero like $1/|\mathbf{x}|$. The second

expression can be written:

$$-\frac{x^k}{|\mathbf{x}|}\int(\mathcal{T}'^{0s}x'^k)_{|s0}\,d\mathbf{x}'+\frac{x^k}{|\mathbf{x}|}\int\mathcal{T}'^{0k}{}_{|0}\,d\mathbf{x}'. \tag{3.2.29}$$

Again the first of the expressions written above vanishes and the only one that must be discussed is:

$$\int\mathcal{T}'^{0k}{}_{|0}\,d\mathbf{x}'. \tag{3.2.30}$$

Now, besides the validity of the integrability condition (3.2.27) we also assume the validity of the Newtonian equations (3.1.4). Therefore, again because $\mathcal{T}^{\alpha\beta}$ vanishes beyond a finite region, we have:

$$\int\mathcal{T}'^{0k}{}_{|0}\,d\mathbf{x}'=\int d\mathbf{x}'\int d\mathbf{x}''\mathcal{T}'^{00}\mathcal{T}''^{00}\left(\frac{1}{|\mathbf{x}'-\mathbf{x}''|}\right)_{|k'}\equiv 0. \tag{3.2.31}$$

We see that $\underset{4}{h}_{00}$ vanishes at infinity but only if the Newtonian equations are fulfilled.

3. THE EQUATIONS OF MOTION IN THE POST-NEWTONIAN APPROXIMATION

We assume that the Newtonian equations are fulfilled. Indeed the zero equation must be fulfilled, because it is a necessary integrability condition. Also the other equations must be fulfilled if we wish $\underset{4}{h}_{00}$ to vanish at infinity. Therefore the post-Newtonian equations of motion may be written:

$$\underset{5}{\mathcal{T}}^{0\beta}{}_{;\beta}=0,\qquad \underset{6}{\mathcal{T}}^{m\beta}{}_{;\beta}=0. \tag{3.3.1}$$

In accordance with what was said in the previous chapter these equations must be understood as equations of the fifth and sixth order for dynamic variables in the post-Newtonian approximation of which $\mathcal{T}^{\alpha\beta}$ is a function. Besides, the $\mathcal{T}^{\alpha\beta}$ may also depend on the metric tensor.

We start by writing out the first of these equations:

$$\underset{5}{\mathcal{T}^{00}}{}_{|0}+\underset{5}{\mathcal{T}^{0s}}{}_{|s}+\underset{3}{\begin{Bmatrix}0\\00\end{Bmatrix}}\underset{2}{\mathcal{T}^{00}}+2\underset{2}{\begin{Bmatrix}0\\0s\end{Bmatrix}}\underset{3}{\mathcal{T}^{0s}}=$$

$$=\underset{5}{\mathcal{T}^{00}}{}_{|0}+\underset{5}{\mathcal{T}^{0s}}{}_{|s}+\tfrac{1}{2}\varphi_{|0}\underset{2}{\mathcal{T}^{00}}+\varphi_{|s}\underset{3}{\mathcal{T}^{0s}}=$$

$$=\Big(\underset{4}{\mathcal{T}^{00}}+\tfrac{1}{2}\varphi\underset{2}{\mathcal{T}^{00}}\Big)_{\underset{1}{|0}}+\Big(\underset{5}{\mathcal{T}^{0s}}+\tfrac{1}{2}\varphi\underset{3}{\mathcal{T}^{0s}}\Big)_{|s}+\tfrac{1}{2}\varphi_{|s}\underset{3}{\mathcal{T}^{0s}}=0 \tag{3.3.2}$$

or, since

$$\varphi=-2\int d\mathbf{x}'\underset{2}{\mathcal{T}'^{00}}|\mathbf{x}-\mathbf{x}'|^{-1}, \tag{3.3.3}$$

we have:

$$\left(\underset{4}{\mathcal{T}^{00}}-\int d\mathbf{x}'\frac{\mathcal{T}'^{00}\mathcal{T}^{00}}{|\mathbf{x}-\mathbf{x}'|}\right)_{|0}+\left(\underset{5}{\mathcal{T}^{0s}}-\int d\mathbf{x}'\frac{\mathcal{T}'^{00}\mathcal{T}^{0s}}{|\mathbf{x}-\mathbf{x}'|}\right)_{|s}-$$

$$-\int d\mathbf{x}'\mathcal{T}'^{00}\mathcal{T}^{0s}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)_{|s}=0. \tag{3.3.4}$$

Now as to the second of equations (3.3.1):

$$\underset{5}{\mathcal{T}^{m0}}{}_{\underset{1}{|0}}+\underset{6}{\mathcal{T}^{ms}}{}_{|s}+\underset{2}{\begin{Bmatrix}m\\00\end{Bmatrix}}\underset{4}{\mathcal{T}^{00}}+\underset{4}{\begin{Bmatrix}m\\00\end{Bmatrix}}\underset{2}{\mathcal{T}^{00}}+2\underset{3}{\begin{Bmatrix}m\\0s\end{Bmatrix}}\underset{3}{\mathcal{T}^{0s}}+\underset{2}{\begin{Bmatrix}m\\rs\end{Bmatrix}}\underset{4}{\mathcal{T}^{rs}}=0. \tag{3.3.5}$$

Here the Christoffel symbols are:

$$\underset{2}{\begin{Bmatrix}m\\00\end{Bmatrix}}=\tfrac{1}{2}\varphi_{|m},\qquad \underset{4}{\begin{Bmatrix}m\\00\end{Bmatrix}}=\tfrac{1}{2}\underset{4}{h}_{00|m}-\underset{3}{h}_{0m\underset{1}{|0}}+\tfrac{1}{2}\varphi\varphi_{|m},$$

$$2\underset{3}{\begin{Bmatrix}m\\0s\end{Bmatrix}}=\underset{3}{h}_{0s|m}-\underset{3}{h}_{0m|s}-\delta_{ms}\varphi_{|0},$$

$$\underset{2}{\begin{Bmatrix}a\\mn\end{Bmatrix}}=\tfrac{1}{2}(\delta_{mn}\varphi_{|a}-\delta_{na}\varphi_{|m}-\delta_{ma}\varphi_{|n}). \tag{3.3.6}$$

In place of the h's we must introduce their expressions (3.2.11) and (3.2.25), then after taking account of the Newtonian equations of motion, we obtain, remembering again that φ stands for an

integral according to (3.2.16):

$$\underset{6}{\mathcal{T}}^{\alpha\beta}{}_{;\beta} = (\underset{5}{\mathcal{T}}^{a0} - \varphi\underset{3}{\mathcal{T}}^{a0} - 4\int d\mathbf{x}'\underset{2}{\mathcal{T}}^{00}\underset{3}{\mathcal{T}}'^{a0}|\mathbf{x}-\mathbf{x}'|^{-1})_{|0} +$$

$$+(\underset{6}{\mathcal{T}}^{ab} - \varphi\underset{4}{\mathcal{T}}^{ab} - 4\int d\mathbf{x}'\underset{3}{\mathcal{T}}^{b0}\underset{3}{\mathcal{T}}'^{a0}|\mathbf{x}-\mathbf{x}'|^{-1})_{|b} +$$

$$+\tfrac{1}{2}\varphi_{|a}(\underset{4}{\mathcal{T}}^{00}+\underset{4}{\mathcal{T}}^{ss}) - \int d\mathbf{x}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)_{|a}[\underset{2}{\mathcal{T}}^{00}(\underset{4}{\mathcal{T}}'^{00}+\underset{4}{\mathcal{T}}'^{ss}) - 4\underset{3}{\mathcal{T}}'^{0s}\underset{3}{\mathcal{T}}^{0s}] +$$

$$+\int d\mathbf{x}'\int d\mathbf{x}''[\mathbf{x},\mathbf{x}',\mathbf{x}'']_{|a}\underset{2}{\mathcal{T}}^{00}\underset{2}{\mathcal{T}}'^{00}\underset{2}{\mathcal{T}}''^{00} - \tfrac{1}{2}\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|_{|a}\underset{4}{\mathcal{T}}'^{00}{}_{|00}\underset{2}{\mathcal{T}}^{00}, \quad (3.3.7)$$

where again

$$[\mathbf{x},\mathbf{x}',\mathbf{x}''] = \frac{1}{|\mathbf{x}-\mathbf{x}'|\,|\mathbf{x}'-\mathbf{x}''|} + \frac{1}{|\mathbf{x}'-\mathbf{x}''|\,|\mathbf{x}''-\mathbf{x}|} + \frac{1}{|\mathbf{x}''-\mathbf{x}|\,|\mathbf{x}-\mathbf{x}'|}. \quad (3.3.8)$$

Since nothing was assumed about the special form of $\mathcal{T}^{\alpha\beta}$, these are the general equations of post-Newtonian motion. We note one interesting feature: $\underset{3}{a}_0$ does not appear in it at all! This is so because it appears in $\underset{3}{h}_{0n}$ and in $\underset{4}{h}_{00}$, but not in the combination $\tfrac{1}{2}\underset{4}{h}_{00|m} - \underset{3}{h}_{0m|0}$ and not in the combination $\underset{3}{h}_{0n|m} - \underset{3}{h}_{0m|n}$ which are the ones that appear in our equations.

Before we descend to the other two levels we would like to bring (3.3.7) into a slightly more convenient form. We see that under the integral sign we have only the $\mathcal{T}$'s with the exception of the last expression in which $\mathcal{T}'^{00}{}_{|00}$ appears. We can, however, change it in the following way:

$$\mathcal{T}'^{00}{}_{|00}\mathcal{T}^{00} = (\mathcal{T}'^{00}{}_{|0}\mathcal{T}^{00})_{|0} - \mathcal{T}'^{00}{}_{|0}\mathcal{T}^{00}{}_{|0}$$

$$= -(\mathcal{T}'^{0s}{}_{|s'}\mathcal{T}^{00})_{|0} - \mathcal{T}'^{0s}{}_{|s'}\mathcal{T}^{0r}{}_{|r}. \quad (3.3.9)$$

The s' derivatives can be shifted (with a change in sign) into the function under the integration sign. A similar procedure, though slightly more complicated, can be carried out with the r derivative.

Thus we obtain:

$$-\tfrac{1}{2}\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|_{|a}\mathcal{T}'^{00}{}_{|00}\mathcal{T}^{00} = -\tfrac{1}{2}\Big(\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|_{|as'}\mathcal{T}'^{0s}\mathcal{T}^{00}\Big)_{|0}+$$
$$-\tfrac{1}{2}\Big(\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|_{|as'}\mathcal{T}'^{0s}\mathcal{T}^{0r}\Big)_{|r}+\tfrac{1}{2}\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|_{|as'r'}\mathcal{T}'^{0s}\mathcal{T}^{0r}. \tag{3.3.10}$$

Therefore, finally, we may write our general equation (2.3.7) in a slightly different form, more convenient for practical use:

$$\Big(\underset{5}{\mathcal{T}}^{a0}-4\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|^{-1}\underset{3}{\mathcal{T}}'^{0a}\underset{2}{\mathcal{T}}^{00}+2\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|^{-1}\underset{2}{\mathcal{T}}'^{00}\underset{3}{\mathcal{T}}^{0a}+$$
$$-\tfrac{1}{2}\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|_{|ab'}\underset{3}{\mathcal{T}}'^{0b}\underset{2}{\mathcal{T}}^{00}\Big)_{|0}+\Big(\underset{6}{\mathcal{T}}^{ab}-4\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|^{-1}\underset{3}{\mathcal{T}}'^{0a}\underset{3}{\mathcal{T}}^{0b}+$$
$$+2\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|^{-1}\underset{2}{\mathcal{T}}'^{00}\underset{4}{\mathcal{T}}^{ab}-\tfrac{1}{2}\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|_{|ac'}\underset{3}{\mathcal{T}}'^{0c}\underset{3}{\mathcal{T}}^{0b}\Big)_{|b}+$$
$$-\int d\mathbf{x}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)_{|a}\big[\underset{2}{\mathcal{T}}'^{00}(\underset{4}{\mathcal{T}}^{00}+\underset{4}{\mathcal{T}}^{ss})+(\underset{4}{\mathcal{T}}'^{00}+\underset{4}{\mathcal{T}}'^{ss})\underset{2}{\mathcal{T}}^{00}-4\underset{3}{\mathcal{T}}'^{0s}\underset{3}{\mathcal{T}}^{0s}\big]+$$
$$+\int d\mathbf{x}'\int d\mathbf{x}''\,[\mathbf{x},\mathbf{x}',\mathbf{x}'']_{|a}\underset{2}{\mathcal{T}}^{00}\underset{2}{\mathcal{T}}'^{00}\underset{2}{\mathcal{T}}''^{00}+$$
$$+\tfrac{1}{2}\int d\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|_{|ab'c}\underset{3}{\mathcal{T}}'^{0b}\underset{3}{\mathcal{T}}^{0c}=0. \tag{3.3.11}$$

Now we wish to descend from the general level to that of continuous and discontinuous distributions. We start with the continuous. According to (3.1.5) our energy-momentum tensor is in this case:

$$\mathcal{T}^{\alpha\beta}=\frac{k}{c^2}\sqrt{-g}\left[\left(\varrho+\frac{\varrho}{c^2}\int\limits_0^p\frac{dp}{\varrho}\right)u^\alpha u^\beta-\frac{p}{c^2}\,g^{\alpha\beta}\right]. \tag{3.3.12}$$

This can be written

$$\mathcal{T}^{\alpha\beta}=\sigma v^\alpha v^\beta-(\eta^{\alpha\beta}+\gamma^{\alpha\beta})\pi \tag{3.3.13}$$

where

$$v^\alpha=\frac{u^\alpha}{u^0}=\frac{dx^\alpha}{dx^0},\quad \sigma=\frac{k}{c^2}\left(\varrho+\frac{\varrho}{c^2}\int\limits_0^p\frac{dp}{\varrho}\right)\sqrt{-g}\,(u^0)^2,\quad \pi=\frac{kp}{c^4}. \tag{3.3.14}$$

We develop (3.3.13) into a power series:

$$\sigma = \underset{2}{\sigma} + \underset{4}{\sigma} + \ldots, \tag{3.3.15a}$$

$$v = \underset{1}{v} + \underset{3}{v} + \ldots, \tag{3.3.15b}$$

$$\pi = \underset{4}{\pi} + \underset{6}{\pi} + \ldots, \tag{3.3.15c}$$

and we obtain:

$$\underset{4}{\mathcal{T}}^{00} = \underset{4}{\sigma} - \underset{4}{\pi}, \qquad \underset{5}{\mathcal{T}}^{0n} = \underset{4}{\sigma}\underset{1}{v}^n + \underset{2}{\sigma}\underset{3}{v}^n,$$

$$\underset{6}{\mathcal{T}}^{mn} = \underset{4}{\sigma}\underset{1}{v}^m\underset{1}{v}^n + \underset{2}{\sigma}\underset{3}{v}^m\underset{1}{v}^n + \underset{2}{\sigma}\underset{1}{v}^m\underset{3}{v}^n + \underset{6}{\pi}\delta^{mn}. \tag{3.3.16}$$

The zeroth equation of motion gives, according to (3.3.4):

$$\Big(\underset{4}{\sigma} - \underset{4}{\pi} - \int d\mathbf{x}'\, |\mathbf{x}-\mathbf{x}'|^{-1} \underset{2}{\sigma}'\underset{2}{\sigma}\Big)_{|0} + \Big(\underset{4}{\sigma}\underset{1}{v}^n + \underset{2}{\sigma}\underset{3}{v}^n - \int d\mathbf{x}'\, |\mathbf{x}-\mathbf{x}'|^{-1} \underset{2}{\sigma}'\underset{2}{\sigma}\underset{1}{v}^n\Big)_{|n} +$$

$$- \int d\mathbf{x}' \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)_{|n} \underset{2}{\sigma}'\underset{2}{\sigma}\underset{1}{v}^n = 0. \tag{3.3.17}$$

Now we put into (3.3.11) the expressions (3.3.16) and we obtain:

$$\Big(\underset{4}{\sigma}\underset{1}{v}^a + \underset{2}{\sigma}\underset{3}{v}^a + 2\int d\mathbf{x}'\, |\mathbf{x}-\mathbf{x}'|^{-1} \underset{2}{\sigma}\underset{2}{\sigma}'(\underset{1}{v}^a - 2\underset{1}{v}'^a) +$$

$$- \tfrac{1}{2}\int d\mathbf{x}'\, |\mathbf{x}-\mathbf{x}'|_{|ab'}\underset{2}{\sigma}\underset{2}{\sigma}'\underset{1}{v}'^b\Big)_{|0} + \Big(\underset{4}{\sigma}\underset{1}{v}^a\underset{1}{v}^b + \underset{2}{\sigma}\underset{3}{v}^a\underset{1}{v}^b + \underset{2}{\sigma}\underset{1}{v}^a\underset{3}{v}^b + \underset{6}{\pi}\delta^{ab} +$$

$$+ 2\int d\mathbf{x}'\, |\mathbf{x}-\mathbf{x}'|^{-1} \underset{2}{\sigma}'\underset{2}{\sigma}\underset{1}{v}^b(\underset{1}{v}^a - 2\underset{1}{v}'^a) - \tfrac{1}{2}\int d\mathbf{x}'\, |\mathbf{x}-\mathbf{x}'|_{|ac'}\underset{2}{\sigma}'\underset{2}{\sigma}\underset{1}{v}'^c\underset{1}{v}^b\Big)_{|b} +$$

$$- \int d\mathbf{x}' \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)_{|a} [\underset{2}{\sigma}'\underset{4}{\sigma} + \underset{2}{\sigma}\underset{4}{\sigma}' - \underset{2}{\sigma}'\underset{4}{\pi} - \underset{2}{\sigma}\underset{4}{\pi}' + \underset{2}{\sigma}'\underset{2}{\sigma}(\underset{1}{v}^s\underset{1}{v}^s + \underset{1}{v}'^s\underset{1}{v}'^s - 4\underset{1}{v}'^s\underset{1}{v}^s)] +$$

$$+ \int d\mathbf{x}' \int d\mathbf{x}''\, [\mathbf{x}, \mathbf{x}', \mathbf{x}'']_{|a}\underset{2}{\sigma}\underset{2}{\sigma}'\underset{2}{\sigma}'' + \tfrac{1}{2}\int d\mathbf{x}'\, |\mathbf{x}-\mathbf{x}'|_{|ab'c}\underset{2}{\sigma}'\underset{2}{\sigma}\underset{1}{v}'^b\underset{1}{v}^c = 0. \tag{3.3.18}$$

These equations determine $\underset{4}{\sigma}$ and $\underset{3}{v}^s$ if π is given as a function of σ and v. They form a transition to the more important problem

of discontinuous particles. Yet the problem of continuous distribution may be of some interest in itself; perhaps in the theory of great nebulas where the internal velocities cannot be assumed to be small compared with the velocity of light; or perhaps in other cosmological applications.

We shall now pass to the more important problem, to the application of the general equations (3.3.4) and (3.3.11) to the case of point particles. Generally we have there:

$$\mathcal{T}^{00} = \sum_{A} \overset{A}{\mu}\,\delta(\mathbf{x}-\overset{A}{\boldsymbol{\xi}}). \tag{3.3.19}$$

If we develop the μ into a power series and the ξ's also, we have, since

$$\underset{4}{\mathcal{T}}^{00} = -\sum_{A} \underset{2}{\overset{A}{\mu}}\,\overset{A}{\delta}_{|s}\,\underset{2}{\overset{A}{\xi}}{}^{s} + \sum_{A} \underset{4}{\overset{A}{\mu}}\,\overset{A}{\delta}, \qquad \overset{A}{\delta} = \delta(\mathbf{x}-\overset{A}{\boldsymbol{\xi}}). \tag{3.3.20}$$

Both here and later we shall leave out the indices written under each quantity if they are the lowest indices. Thus, for instance, μ stands for $\underset{2}{\mu}$ or, permitting an error of a higher order, for $\underset{2}{\mu}+\underset{4}{\mu}$. We have:

$$\underset{4}{\mathcal{T}}^{00} = \sum_{C} (\underset{4}{\overset{C}{\mu}}\,\overset{C}{\delta} - \overset{C}{\mu}\,\overset{C}{\delta}_{|s}\,\underset{2}{\overset{C}{\xi}}{}^{s}), \tag{3.3.21a}$$

$$\underset{5}{\mathcal{T}}^{0n} = \sum_{C} (\underset{4}{\overset{C}{\mu}}\,\overset{C}{\delta}\,\overset{C}{\xi}{}^{n}{}_{|0} - \overset{C}{\mu}\,\overset{C}{\delta}_{|s}\,\underset{2}{\overset{C}{\xi}}{}^{s}\,\overset{C}{\xi}{}^{n}{}_{|0} + \overset{C}{\mu}\,\overset{C}{\delta}\,\underset{3}{\overset{C}{\xi}}{}^{n}{}_{|0}), \tag{3.3.21b}$$

$$\underset{6}{\mathcal{T}}^{mn} = \sum_{C} (\underset{4}{\overset{C}{\mu}}\,\overset{C}{\delta}\,\overset{C}{\xi}{}^{m}{}_{|0}\,\overset{C}{\xi}{}^{n}{}_{|0} + \overset{C}{\mu}\,\overset{C}{\delta}\,\underset{3}{\overset{C}{\xi}}{}^{m}{}_{|0}\,\overset{C}{\xi}{}^{n}{}_{|0} +$$
$$+ \overset{C}{\mu}\,\overset{C}{\delta}\,\overset{C}{\xi}{}^{m}{}_{|0}\,\underset{3}{\overset{C}{\xi}}{}^{n}{}_{|0} - \overset{C}{\mu}\,\overset{C}{\delta}_{|s}\,\underset{2}{\overset{C}{\xi}}{}^{s}\,\overset{C}{\xi}{}^{m}{}_{|0}\,\overset{C}{\xi}{}^{n}{}_{|0}). \tag{3.3.21c}$$

Introducing this into (3.3.2) and (3.3.7) we wish to find

$$\int_{\overset{A}{\Omega}} \underset{5}{\mathcal{T}}^{0\alpha}{}_{;\alpha}\,d\mathbf{x} \quad \text{and} \quad \int_{\overset{A}{\Omega}} \underset{6}{\mathcal{T}}^{m\alpha}{}_{;\alpha}\,d\mathbf{x} \tag{3.3.22}$$

where $\overset{A}{\Omega}$ is a small region surrounding the A'th particle. Therefore we are faced with integrals of the type

$$\int\limits_{\overset{A}{\Omega}} d\mathbf{x} \int\limits_{\Omega} d\mathbf{x}' f(\mathbf{x}, \mathbf{x}')\, \delta(\mathbf{x} - \overset{B}{\boldsymbol{\xi}})\, \delta(\mathbf{x}' - \overset{C}{\boldsymbol{\xi}}). \tag{3.3.23}$$

Since Ω is the entire region, the result is:

$$f(\overset{B}{\boldsymbol{\xi}}, \overset{C}{\boldsymbol{\xi}})\, \delta_{AB}. \tag{3.3.24}$$

Thus this is the simple general rule of calculating the integrals (3.2.23). But special attention must be paid to the case $B = C$. Namely if f is a singular function, e. g.

$$f = |\mathbf{x} - \mathbf{x}'|^{-1}, \tag{3.3.25}$$

then we have an expression

$$|\overset{A}{\boldsymbol{\xi}} - \overset{B}{\boldsymbol{\xi}}|^{-1} = r_{AB}^{-1} = \infty \qquad \text{for} \qquad A = B. \tag{3.3.26}$$

But it is here that our argument about the choice of good δ's intervenes. They annihilate this type of contribution. However, if $f(\overset{A}{\boldsymbol{\xi}}, \overset{B}{\boldsymbol{\xi}})$ is finite for $A = B$, then, of course, it must be taken into account. This is especially true in expressions of the type

$$\sum_{B,C} \frac{\overset{A}{\mu}\overset{B}{\mu}\overset{C}{\mu}}{r_{AB} r_{AC}}. \tag{3.3.27}$$

Here we see, for example, that the expressions $B = C$ must not be ignored, and we obtain:

$$\sum_{B,C} \frac{\overset{A}{\mu}\overset{B}{\mu}\overset{C}{\mu}}{r_{AB} r_{AC}} = \sum_{B,C}{}' \frac{\overset{A}{\mu}\overset{B}{\mu}\overset{C}{\mu}}{r_{AB} r_{AC}} + \sum_{B}{}' \frac{\overset{A}{\mu}\overset{B}{\mu}{}^{2}}{r_{AB}^{2}}. \tag{3.3.28}$$

where Σ' means that the sum is not to be taken for $B = C$ or $A = B$ or $A = C$.

After these remarks the calculation is fairly straightforward. We start by calculating $\overset{A}{\underset{4}{\mu}}$ from (3.3.2). Integrated over $\overset{A}{\Omega}$, because of the Newtonian equations of motion, and because the divergence expression does not give any contribution, this expression gives:

$$\overset{A}{\underset{4}{\mu}} = {\sum_B}' \frac{\overset{AB}{\mu\mu}}{r_{AB}} + \tfrac{1}{2}\overset{AA}{\mu\xi^s}{}_{|0}\overset{A}{\xi^s}{}_{|0}. \tag{3.3.29}$$

We must put this expression into (3.3.21). (The arbitrary additive constant can be absorbed by $\underset{2}{\mu}$.) The rest is a straightforward calculation which finally gives us:

$$\begin{aligned}
&\Big[\overset{AA}{\mu\underset{3}{\xi}{}^a}{}_{|0} + \Big(\tfrac{1}{2}\overset{AA}{\mu\xi^s}{}_{|0}\overset{A}{\xi^s}{}_{|0} + 3{\sum_B}' \overset{AB}{\mu\mu} r_{AB}^{-1}\Big)\overset{A}{\xi^a}{}_{|0} + \\
&-{\sum_B}' 4\overset{AB}{\mu\mu} r_{AB}^{-1}\overset{B}{\xi^a}{}_{|0} - \tfrac{1}{2}{\sum_B}' \overset{AB}{\mu\mu} r_{AB|\overset{A}{\xi^a}\overset{B}{\xi^s}}\overset{B}{\xi^s}{}_{|0}\Big]_{|0} + \\
&-{\sum_B}' \overset{AB}{\mu\mu}\big[\overset{A}{\underset{2}{\xi}{}^s}(r_{AB}^{-1})_{|\overset{A}{\xi^a}\overset{A}{\xi^s}} + \overset{B}{\underset{2}{\xi}{}^s}(r_{AB}^{-1})_{|\overset{A}{\xi^a}\overset{B}{\xi^s}}\big] + \\
&-{\sum_B}' \overset{AB}{\mu\mu}\big(\tfrac{3}{2}\overset{A}{\xi^s}{}_{|0}\overset{A}{\xi^s}{}_{|0} + \tfrac{3}{2}\overset{B}{\xi^s}{}_{|0}\overset{B}{\xi^s}{}_{|0} - 4\overset{A}{\xi^s}{}_{|0}\overset{B}{\xi^s}{}_{|0}\big)(r_{AB}^{-1})_{|\overset{A}{\xi^a}} + \\
&+\tfrac{1}{2}{\sum_B}' \overset{ABA}{\mu\mu\xi^r}{}_{|0}\overset{B}{\xi^s}{}_{|0} r_{AB|\overset{A}{\xi^a}\overset{B}{\xi^s}\overset{A}{\xi^r}} + \tfrac{1}{2}{\sum_B}' \overset{AB}{\mu\mu}(\overset{A}{\mu}+\overset{B}{\mu})(r_{AB}^{-2})_{|\overset{A}{\xi^a}} + \\
&+\tfrac{1}{2}{\sum_{B,C}}' \overset{ABC}{\mu\mu\mu}\Big(\frac{1}{r_{AB}}\frac{1}{r_{BC}} + \frac{1}{r_{BC}}\frac{1}{r_{CA}} + \frac{1}{r_{CA}}\frac{1}{r_{AB}}\Big)_{|\overset{A}{\xi^a}} = 0.
\end{aligned} \tag{3.3.30}$$

These are the final equations of motion for N particles in the post-Newtonian approximation.

Obviously in the expressions of the sixth order which do not contain the $\underset{2}{\xi}$'s it is immaterial whether we take $\overset{A}{\xi^s}$ or $\overset{A}{\xi^s} + \overset{A}{\underset{2}{\xi}{}^s}$. The $\overset{A}{\underset{2}{\xi}{}^s}$

appear only in the expressions

$$\underset{4}{\overset{A}{\mu}}\overset{A}{\xi}{}^{a}{}_{|00}-\sum_{B}{}' \overset{A}{\mu}\overset{B}{\mu}\Big[\underset{2}{\overset{A}{\xi}}{}^{s}(r_{AB}^{-1})_{|\overset{A}{\xi}{}^{s}\overset{A}{\xi}{}^{a}}+\underset{2}{\overset{B}{\xi}}{}^{s}(r_{AB}^{-1})_{|\overset{A}{\xi}{}^{a}\overset{B}{\xi}{}^{s}}\Big]. \tag{3.3.31}$$

All other expressions are already known from the Newtonian approximation. Therefore we have $3N$ differential equations to determine the $3N$ unknown $\underset{2}{\xi}$'s.

We mentioned before the difference between the equations of motion of a certain approximation and those up to a certain approximation. Here we followed the theory and practice of equations of motion of a certain approximation. The theory up to a certain approximation requires the use of dipole fields. However, as we also mentioned before, the story is essentially simpler if we do not go further than the post-Newtonian stage. There the only integrability condition that we met was

$$\underset{5}{\mathcal{T}}{}^{0\beta}{}_{;\beta}=0 \tag{3.3.32}$$

which defines $\underset{4}{\mu}$. Thus with the proper choice of $\underset{4}{\mu}$ this integrability condition is fulfilled. Therefore, ignoring the Newtonian equations of motion, we can write the equations of motion up to the post-Newtonian order. They are:

$$\underset{4}{\mathcal{T}}{}^{m\beta}{}_{;\beta}+\underset{6}{\mathcal{T}}{}^{m\beta}{}_{;\beta}=0. \tag{3.3.33}$$

Here we have the unknown functions:

$$\overset{A}{\xi}{}^{s}=\underset{0}{\overset{A}{\xi}}{}^{s}+\underset{2}{\overset{A}{\xi}}{}^{s} \tag{3.3.34}$$

and by integrating (3.3.33), we obtain at once the motion up to the post-Newtonian order. Since in $\underset{6}{\mathcal{T}}{}^{m\beta}{}_{;\beta}$ it is immaterial whether we put $\overset{A}{\xi}{}^{s}$ or $\underset{0}{\overset{A}{\xi}}{}^{s}$, we can regard this expression as known if the solution of the Newtonian equations is known to us. Thus we can write down

the equations of motion up to the sixth order, introducing

$$\underset{2}{\mathcal{T}}^{00}+\underset{4}{\mathcal{T}}^{00} = \sum_{A} \overset{A}{\underset{2}{\mu}}\overset{A}{\delta}+\sum_{A} \overset{A}{\underset{4}{\mu}}\overset{A}{\delta}, \qquad \overset{A}{\delta} = \delta(\mathbf{x}-\overset{A}{\boldsymbol{\xi}}).$$

Then we have:

$$\Big[\Big(\overset{A}{\mu}+\tfrac{1}{2}\overset{A}{\mu}\overset{A}{\xi}{}^{s}{}_{|0}\overset{A}{\xi}{}^{s}{}_{|0}+3\sum_{B}{}' \overset{A}{\mu}\overset{B}{\mu} r^{-1}_{AB}\Big)\overset{A}{\xi}{}^{a}{}_{|0}-\tfrac{1}{2}\sum_{B}{}' \overset{A}{\mu}\overset{B}{\mu} r_{AB|\overset{A}{\xi}{}^{a}\overset{B}{\xi}{}^{s}}\overset{B}{\xi}{}^{s}{}_{|0}+$$

$$-4\sum_{B}{}' \overset{A}{\mu}\overset{B}{\mu} r^{-1}_{AB}\overset{B}{\xi}{}^{a}{}_{|0}\Big]_{|0}-\sum_{B}{}' \overset{A}{\mu}\overset{B}{\mu}\big(1+\tfrac{3}{2}\overset{A}{\xi}{}^{s}{}_{|0}\overset{A}{\xi}{}^{s}{}_{|0}+\tfrac{3}{2}\overset{B}{\xi}{}^{s}{}_{|0}\overset{B}{\xi}{}^{s}{}_{|0}-4\overset{A}{\xi}{}^{s}{}_{|0}\overset{B}{\xi}{}^{s}{}_{|0}\big)$$

$$(r^{-1}_{AB})_{|\overset{A}{\xi}{}^{a}}+\tfrac{1}{2}\sum_{B}{}' \overset{A}{\mu}\overset{B}{\mu}\overset{A}{\xi}{}^{r}{}_{|0}\overset{B}{\xi}{}^{s}{}_{|0} r_{AB|\overset{A}{\xi}{}^{a}\overset{B}{\xi}{}^{s}\overset{A}{\xi}{}^{r}}+\tfrac{1}{2}\sum_{B}{}' \overset{A}{\mu}\overset{B}{\mu}(\overset{A}{\mu}+\overset{B}{\mu})(r^{-2}_{AB})_{|\overset{A}{\xi}{}^{a}}+$$

$$+\tfrac{1}{2}\sum_{B,C}{}' \overset{A}{\mu}\overset{B}{\mu}\overset{C}{\mu}\left(\frac{1}{r_{AB}}\frac{1}{r_{BC}}+\frac{1}{r_{BC}}\frac{1}{r_{CA}}+\frac{1}{r_{CA}}\frac{1}{r_{AB}}\right)_{|\overset{A}{\xi}{}^{a}} = 0. \tag{3.3.35}$$

These equations can be derived from a Lagrangian. Indeed, these expressions have the form

$$(L_{|\overset{A}{\xi}{}^{a}{}_{|0}})_{|0}-L_{|\overset{A}{\xi}{}^{a}} = 0 \tag{3.3.36}$$

where $L = \underset{4}{L}+\underset{6}{L}$ and

$$L = \tfrac{1}{2}\sum_{A} \overset{A}{\mu}\overset{A}{\xi}{}^{s}{}_{|0}\overset{A}{\xi}{}^{s}{}_{|0}+\tfrac{1}{8}\sum_{A} \overset{A}{\mu}(\overset{A}{\xi}{}^{s}{}_{|0}\overset{A}{\xi}{}^{s}{}_{|0})^{2}+$$

$$+\tfrac{3}{4}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu} r^{-1}_{AB}(\overset{A}{\xi}{}^{s}{}_{|0}\overset{A}{\xi}{}^{s}{}_{|0}+\overset{B}{\xi}{}^{s}{}_{|0}\overset{B}{\xi}{}^{s}{}_{|0})-2\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu} r^{-1}{}_{AB}\overset{A}{\xi}{}^{s}{}_{|0}\overset{B}{\xi}{}^{s}{}_{|0}+$$

$$-\tfrac{1}{4}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu} r_{AB|\overset{A}{\xi}{}^{c}\overset{B}{\xi}{}^{b}}\overset{B}{\xi}{}^{b}{}_{|0}\overset{A}{\xi}{}^{c}{}_{|0}+\tfrac{1}{2}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu} r^{-1}_{AB}-\tfrac{1}{4}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}(\overset{A}{\mu}+\overset{B}{\mu})r^{-2}_{AB}+$$

$$-\tfrac{1}{6}\sum_{A,B,C}{}' \overset{A}{\mu}\overset{B}{\mu}\overset{C}{\mu}(r^{-1}_{AB}r^{-1}_{BC}+r^{-1}_{BC}r^{-1}_{CA}+r^{-1}_{CA}r^{-1}_{AB}). \tag{3.3.37}$$

Obviously this is not a satisfactory way of obtaining a Lagrangian

since its existence appears accidental and is not connected with the theory. We shall come back to this problem in the next chapter.

There is one more question which we should like to discuss briefly and generally without going into particular calculations. While finding the equations of motion we have used our good δ's and therefore we did not have any contributions from the self-terms. This was because, according to our rules, we put

$$\omega_{AA} = \int_{\overset{A}{\Omega}} d\mathbf{x} \int d\mathbf{x}' \, |\mathbf{x}-\mathbf{x}'|^{-1} \, \delta(\mathbf{x}-\overset{A}{\xi}) \, \delta(\mathbf{x}'-\overset{A}{\xi}) = 0. \qquad (3.3.38)$$

The question arises: would the result have been changed if ω_{AA} were not set equal to zero? To find a rigorous answer to this question we should have to investigate in detail the transition from a continuous to a discontinuous distribution. This is a rather troublesome and lengthy procedure which we shall therefore omit here. Its result is almost obvious. Let us assume that we start from drops of dimension l. Then the post-Newtonian equations of motion of the center of mass of each of these drops are, with the accuracy $O(l)$, identical with those derived here in (3.3.35) or in (3.3.33). For $l \to 0$, there are some additive singular expressions too, of the order $O(l^{-1})$. However, they give a contribution only to the mass of order μ^2/l. Since μ is the gravitational radius of this drop, then we obtain a correction to the mass of the order

$$\mu\left(1+\frac{\text{gravitational radius}}{\text{dimensions}}\right). \qquad (3.3.39)$$

This quantity is not observable, because it appears only as an additional constant.

Coming back to the question of what would have happened if ω_{AA} were not equal to zero, or, in other words, if we had used different δ's and not the good ones, then the answer is: we would have to use the renormalization procedure in a way that is similar to its use in quantum electrodynamics. Since it can be shown that the theory can be renormalized, we see that the use of our good δ's avoids the process of renormalization.

4. THE CONSERVATION LAWS FOR A SYSTEM OF PARTICLES

From the existence of a Lagrangian

$$L = \underset{4}{L} + \underset{6}{L} \tag{3.4.1}$$

and from its invariance with respect to certain transformations, we can deduce certain conservation laws. And so:

The conservation of gravitational energy E can be deduced from the invariance of the Lagrangian L with respect to the transformation

$$x'^0 = x^0 + a^0, \qquad a^0 = \text{const.} \tag{3.4.2}$$

As a definition of the constant E, we obtain:

$$\sum_A \overset{A}{\xi}{}^s{}_{|0} L_{|\overset{A}{\xi}{}^s{}_{|0}} - L = E.$$

Introducing (3.3.37) here in place of L, we obtain:

$$\begin{aligned}
&\tfrac{1}{2}\sum_{A} \overset{A}{\mu}\overset{A}{\xi}{}^s{}_{|0}\overset{A}{\xi}{}^s{}_{|0} - \tfrac{1}{2}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu} r_{AB}^{-1} + \tfrac{3}{8}\sum_{A} \overset{A}{\mu}(\overset{A}{\xi}{}^s{}_{|0}\overset{A}{\xi}{}^s{}_{|0})^2 + \\
&+\tfrac{3}{4}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}(\overset{A}{\xi}{}^s{}_{|0}\overset{A}{\xi}{}^s{}_{|0} + \overset{B}{\xi}{}^s{}_{|0}\overset{B}{\xi}{}^s{}_{|0}) r_{AB}^{-1} - 2\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}\overset{A}{\xi}{}^s{}_{|0}\overset{B}{\xi}{}^s{}_{|0} r_{AB}^{-1} + \\
&-\tfrac{1}{4}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu} r_{AB|\overset{A}{\xi}{}^s\overset{B}{\xi}{}^r}\overset{A}{\xi}{}^s{}_{|0}\overset{B}{\xi}{}^r{}_{|0} + \tfrac{1}{4}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}(\overset{A}{\mu} + \overset{B}{\mu}) r_{AB}^{-1} + \\
&+\tfrac{1}{6}\sum_{A,B,C}{}' \overset{A}{\mu}\overset{B}{\mu}\overset{C}{\mu}(r_{AB}^{-1} r_{BC}^{-1} + r_{BC}^{-1} r_{CA}^{-1} + r_{CA}^{-1} r_{AB}^{-1}) = \underset{\to 6}{E}.
\end{aligned} \tag{3.4.3}$$

Up to the fourth order we find the Newtonian conservation law:

$$\sum_A \overset{A}{\mu} + \tfrac{1}{2}\sum_A \overset{A}{\mu}\overset{A}{\xi}{}^s{}_{|0}\overset{A}{\xi}{}^s{}_{|0} - \tfrac{1}{2}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu} r_{AB}^{-1} = \underset{\to 4}{E}. \tag{3.4.4}$$

if we add $\sum_A \overset{A}{\mu} = \underset{2}{E}$ as the constant of integration.

The conservation of gravitational linear momentum can be deduced from the invariance with respect to the transformation

$$\overset{A}{\xi}{}^a \to \overset{A}{\xi}{}^a + a^a, \qquad a^a = \text{const.} \tag{3.4.5}$$

As a definition of the constant P^k we obtain:

$$\sum_A L_{|\overset{A}{\xi}{}^k_{|0}} = P^k.$$

Introducing the proper Lagrangian, we obtain:

$$\sum_A \Big(1 + \tfrac{1}{2}\overset{A}{\xi}{}^s_{|0}\overset{A}{\xi}{}^s_{|0} - \tfrac{1}{2}\sum_B{}' \overset{B}{\mu} r^{-1}_{AB}\Big)\overset{A}{\mu}\overset{A}{\xi}{}^a_{|0} +$$

$$-\tfrac{1}{2}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}(\overset{A}{\xi}{}^a - \overset{B}{\xi}{}^a)(\overset{A}{\xi}{}^b - \overset{B}{\xi}{}^b)\overset{B}{\xi}{}^b_{|0} r^{-3}_{AB} = \underset{\to 5}{P^a}. \tag{3.4.6}$$

The conservation of gravitational rotational momentum $J^{[ab]}$ can be deduced from the invariance of L with respect to the transformation

$$\xi^a \to M^a{}_b\, \xi^b$$

where $M^a{}_b$ is a constant orthogonal transformation matrix. We obtain as a definition of the constant $J^{[ab]}$

$$\sum_A \Big(L_{|\overset{A}{\xi}{}^a_{|0}}\overset{A}{\xi}{}^b - L_{|\overset{A}{\xi}{}^b_{|0}}\overset{A}{\xi}{}^a\Big) = \underset{\to 5}{J^{[ab]}}. \tag{3.4.7}$$

Introducing the proper Lagrangian, we obtain:

$$\sum_A \overset{A}{\mu}\Big(1 + 3\sum_B{}' \overset{B}{\mu} r^{-1}_{AB} + \tfrac{1}{2}\overset{A}{\xi}{}^s_{|0}\overset{A}{\xi}{}^s_{|0}\Big)(\overset{A}{\xi}{}^a_{|0}\overset{A}{\xi}{}^b - \overset{A}{\xi}{}^b_{|0}\overset{A}{\xi}{}^a) +$$

$$-4\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}(\overset{B}{\xi}{}^a_{|0}\overset{A}{\xi}{}^b - \overset{B}{\xi}{}^b_{|0}\overset{A}{\xi}{}^a) r^{-1}_{BA} +$$

$$+\tfrac{1}{2}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}\overset{B}{\xi}{}^c_{|0}\big(r_{AB|\overset{A}{\xi}{}^a\overset{B}{\xi}{}^c}\overset{A}{\xi}{}^b - r_{AB|\overset{A}{\xi}{}^b\overset{B}{\xi}{}^c}\overset{A}{\xi}{}^a\big) = \underset{\to 5}{J^{[ab]}}. \tag{3.4.8}$$

We obtain the conservation of the gravitational mass center in a slightly different way, since L is not invariant with respect to a Galileo transformation. Because of Newton's equations of motion, we have from (3.4.6):

$$\frac{d}{dx^0}\left(\sum_{A}\overset{A}{\mu}\overset{A}{\xi}{}^{a}\left(1+\tfrac{1}{2}\overset{A}{\xi}{}^{b}{}_{|0}\overset{A}{\xi}{}^{b}{}_{|0}-\tfrac{1}{2}\sum_{B}{}'\overset{B}{\mu}r^{-1}_{AB}\right)\right)=\underset{\to 5}{P^{a}} \tag{3.4.9}$$

from which follows:

$$\sum_{A}\overset{A}{\mu}\overset{A}{\xi}{}^{a}\left(1+\tfrac{1}{2}\overset{A}{\xi}{}^{b}{}_{|0}\overset{A}{\xi}{}^{b}{}_{|0}-\tfrac{1}{2}\sum_{B}{}'\overset{B}{\mu}r^{-1}_{AB}\right)=\underset{\to 5}{P^{a}}x^{0}+Q^{a} \tag{3.4.10}$$

where Q^a is constant.

There are some interesting physical conclusions that can be deduced from these equations. We see that the gravitational total mass as defined by the conservation laws is different from the total inertial mass, as previously defined. Indeed, let us call

$\sum_{A}(\underset{2}{\overset{A}{\mu}}+\underset{4}{\overset{A}{\mu}})$ — the total inertial mass up to the fourth order,

$\underset{4}{E}$ — the total gravitational mass up to the fourth order.

Then we have, as we see from (3.3.29) and (3.4.3):

$$\sum_{A}\underset{4}{\overset{A}{\mu}}=\tfrac{1}{2}\sum_{A}\overset{A}{\mu}\overset{A}{\xi}{}^{s}{}_{|0}\overset{A}{\xi}{}^{s}{}_{|0}+\sum_{A,B}\frac{\overset{AB}{\mu\mu}}{r_{AB}}=\underset{4}{\mu}{}_{(IN)},$$

$$\tfrac{1}{2}\sum_{A}\overset{A}{\mu}\overset{A}{\xi}{}^{s}{}_{|0}\overset{A}{\xi}{}^{s}{}_{|0}-\tfrac{1}{2}\sum_{A,B}{}'\frac{\overset{AB}{\mu\mu}}{r_{AB}}=\underset{4}{\mu}{}_{(G)}=\underset{4}{E}. \tag{3.4.11}$$

It is the total gravitational, and not the total inertial mass, that is conserved. It is the center of gravity of this gravitational, and not inertial, mass which moves uniformly.

We shall return to this problem in the last chapter. Then we shall also discuss an additional, more general problem: can we, independ-

ently of the Lagrangian and of the approximation method, formulate the conservation laws in a field language, for an arbitrary energy-momentum tensor. What this reasoning has shown us is the intimate connection between these conservation laws and the equations of motion. But the equations of motion can be expressed in a field language. The same must be possible for the conservation laws. It is this problem to which we shall return in the last chapter.

CHAPTER IV

THE VARIATIONAL PRINCIPLE AND THE EQUATIONS OF MOTION OF THE THIRD KIND

1. FORMULATION OF THE PROBLEM

In Section 3 of Chapter I, we discussed the equations of motion of the first and second kinds starting from a variational principle. Let us recall briefly how we formulated the equations of motion of the second kind.

We introduced

$$ds_A = (\overset{A}{\tilde{g}}_{\alpha\beta} d\overset{A}{\xi}{}^{\alpha} d\overset{A}{\xi}{}^{\beta})^{1/2} \tag{4.1.1}$$

where

$$\overset{A}{\tilde{g}}_{\alpha\beta} = \int d\mathbf{x}\, \delta(\mathbf{x} - \overset{A}{\boldsymbol{\xi}}) g_{\alpha\beta} \tag{4.1.2}$$

and then we varied the integral

$$W' = -\sum_{A=1}^{N} \overset{A}{m}_{(0)} c \int_{\sigma_1}^{\sigma_2} (\overset{A}{\tilde{g}}_{\alpha\beta} d\overset{A}{\xi}{}^{\alpha} d\overset{A}{\xi}{}^{\beta})^{1/2} \tag{4.1.3}$$

with respect to ξ^{α}, where $\delta\xi^{\alpha}$ vanished at the ends of the interval. The result was

$$\sum_{A=1}^{N} \overset{A}{m}_{(0)} c \int_{\sigma_1}^{\sigma_2} \delta\overset{A}{\xi}{}^{\alpha} \left(\frac{d}{ds_A} \overset{A}{\tilde{g}}_{\alpha\beta} \frac{d\overset{A}{\xi}{}^{\beta}}{ds_A} - \tfrac{1}{2} \overset{A}{\widetilde{g_{\mu\nu|\alpha}}} \frac{d\overset{A}{\xi}{}^{\mu}}{ds_A} \frac{d\overset{A}{\xi}{}^{\nu}}{ds_A} \right), \tag{4.1.4}$$

from which the equations of motion could easily be deduced.

All this has already been done. Here we should like to draw attention to one essential assumption which accompanied this reasoning. It was: the $g_{\alpha\beta}$ have to be treated as given functions, not to be varied with respect to the ξ's. This means: using the definition (4.1.2) and varying $\tilde{g}_{\alpha\beta}$ with respect to the ξ's, we only took into account its dependence on the ξ's through the δ's. But in fact we know that the g's depend on the ξ's and their time derivatives:

$$g_{\alpha\beta} = g_{\alpha\beta}(x^0, x^k, \overset{1}{\xi}{}^k, \overset{1}{\xi}{}^k{}_{|0}, \dots, \overset{N}{\xi}{}^k, \overset{N}{\xi}{}^k{}_{|0}) \tag{4.1.5}$$

if we assume, as we do here, that only the first time derivatives of ξ^k enter into the g's.

We remember also that the tweedling process consisted in replacing the x's by the ξ's and getting rid of the singularities. Thus $\overset{A}{\tilde{g}}_{\alpha\beta}$ meant:

$$\overset{A}{\tilde{g}}_{\alpha\beta} = g_{\alpha\beta}(x^0, \overset{A}{\xi}{}^k, \overset{1}{\xi}{}^k, \overset{1}{\xi}{}^k{}_{|0}, \dots, \overset{N}{\xi}{}^k, \overset{N}{\xi}{}^k{}_{|0}) . \tag{4.1.6}$$

Also we recall from (0.18), or from what is said in Appendix 2:

$$\overset{A}{\tilde{g}}_{\alpha\beta|0} = \overbrace{g_{\alpha\beta|0}}^{A} + \overbrace{g_{\alpha\beta|s}}^{A} \overset{A}{\xi}{}^s{}_0 = \overbrace{g_{\alpha\beta|\nu}}^{A} \overset{A}{\xi}{}^\nu{}_{|0} , \tag{4.1.7}$$

$$\overset{A}{\tilde{g}}_{\alpha\beta|\overset{B}{\xi}{}^s} = \overbrace{g_{\alpha\beta|\overset{B}{\xi}{}^s}}^{A} + \overbrace{g_{\alpha\beta|s}}^{A} \delta_{AB} . \tag{4.1.8}$$

We can now formulate our problem. We are looking for a Lagrangian

$$L = L(x^0, \overset{1}{\xi}{}^k, \overset{1}{\xi}{}^k{}_{|0}, \dots, \overset{N}{\xi}{}^k, \overset{N}{\xi}{}^k{}_{|0}) \tag{4.1.9}$$

such that a variation of the action with respect to the ξ's, vanishing at the end of the interval of integration, would give the proper equations of motion of the third kind, that is

$$\delta \int_{x'^0}^{x''^0} L dx^0 = 0 \tag{4.1.10}$$

would give the equations of motion of all the particles.

Such a variational principle, if it exists, we shall call Fokker's variational principle since he was the first to introduce it into electrodynamics.

It would seem that a good way to start would be to imitate the way we obtained the right equations of motion of the second kind. Let us therefore, as the first attempt, start with a Lagrangian L':

$$L' = L'(x^0, \xi^k, \xi^k{}_{|0}) = -\sum_A \overset{A}{m}_{(0)} c \big(\overset{A}{\widetilde{g}}_{\alpha\beta} \overset{A}{\xi}{}^{\alpha}{}_{|0} \overset{A}{\xi}{}^{\beta}{}_{|0}\big)^{1/2}. \tag{4.1.11}$$

Here, the $\tilde{g}_{\alpha\beta}$'s have to be treated as functions of the ξ's, the $\xi_{|0}$'s and x^0.

We form the equations of motion from such a Lagrangian:

$$L'_{\big|\overset{B}{\xi}{}^k} - \big(L'_{\big|\overset{B}{\xi}{}^k{}_{|0}}\big)_{|0} = 0. \tag{4.1.12}$$

If, as before, we introduce:

$$\overset{A}{\mu} = \frac{k\overset{A}{m}_{(0)}}{c^2} \frac{dx^0}{ds_A} = \frac{k\overset{A}{m}_{(0)}}{c^2} \big(\overset{A}{\widetilde{g}}_{\alpha\beta} \overset{A}{\xi}{}^{\alpha}{}_{|0} \overset{A}{\xi}{}^{\beta}{}_{|0}\big)^{-1/2} \tag{4.1.13}$$

this gives us the following equation, because of (4.1.8):

$$\begin{aligned}
&\big(\overset{B}{\mu} \overset{B}{\widetilde{g}}_{s\beta} \overset{B}{\xi}{}^{\beta}{}_{|0}\big)_{|0} - \tfrac{1}{2} \overset{B}{\mu} \overset{B}{\widetilde{g}}_{\alpha\beta|s} \overset{B}{\xi}{}^{\alpha}{}_{|0} \overset{B}{\xi}{}^{\beta}{}_{|0} + \\
&\qquad -\tfrac{1}{2} \sum_A \overset{A}{\mu} \overset{A}{\widetilde{g}}_{\alpha\beta|\overset{B}{\xi}{}^s} \overset{A}{\xi}{}^{\alpha}{}_{|0} \overset{A}{\xi}{}^{\beta}{}_{|0} + \tfrac{1}{2} \Big(\sum_A \overset{A}{\mu} \overset{A}{\widetilde{g}}_{\alpha\beta|\overset{B}{\xi}{}^s{}_{|0}} \overset{A}{\xi}{}^{\alpha}{}_{|0} \overset{B}{\xi}{}^{\beta}{}_{|0}\Big)_{|0} = 0.
\end{aligned} \tag{4.1.14}$$

We must distinguish here between the expressions in the first and in the second row. If there were only expressions of the first row, everything would be splendid. We would then have obtained the three equations of motion written under (1.7.8), the fourth being the definition of $\overset{A}{\mu}$, as expressed in (4.1.13). But the disturbing factor is the two expressions below. Therefore L' is not the proper Lagrangian of the Fokker type. Of course we do not know before-

hand even whether such a Lagrangian exists. For example, in the case of electrodynamics we know that such a Lagrangian does not exist for advanced or retarded action. Thus our aim is to find out whether and under what conditions we can get rid of the expressions written out in the second line of (4.1.14).

If we wish the Fokker Lagrangian to be invariant, or rather $\delta(L\,dx^0)$ to be an invariant, and also to depend only on x^0, the ξ's and the $\xi_{|0}$'s, then there is little choice in this matter. Let us assume, as we did before, that

$$g_{\alpha\beta} = g_{\alpha\beta}(x^0, x^k, \xi^a, \xi^a{}_{|0}). \tag{4.1.15}$$

Since we wish to have only second derivatives of $\xi^k(x^0)$ in the equations of motion, we should avoid higher derivatives of $g_{\alpha\beta}$ as much as possible. Thus our choice is essentially restricted to

$$L'' = \int\limits_{\Omega_{(3)}} \sqrt{-g}\,G\,d\mathbf{x}, \tag{4.1.16}$$

$$G = g^{\mu\nu}\left(\begin{Bmatrix}\varrho\\ \nu\sigma\end{Bmatrix}\begin{Bmatrix}\sigma\\ \mu\varrho\end{Bmatrix} - \begin{Bmatrix}\varrho\\ \mu\nu\end{Bmatrix}\begin{Bmatrix}\sigma\\ \varrho\sigma\end{Bmatrix}\right), \tag{4.1.17}$$

where the integral is to be taken over the entire three-dimensional space. If we write down the variational principle

$$\delta \int\limits_{x'^0}^{x''^0} dx^0 \int\limits_{\Omega_{(3)}} \sqrt{-g}\,G\,d\mathbf{x} = 0 \tag{4.1.18}$$

we see that the left-hand side is an invariant, since $\sqrt{-g}\,G$ differs from $\sqrt{-g}\,R$ only by expressions that can be put into the form of a (four-dimensional) divergence.

First we vary $\sqrt{-g}\,G$ with respect to $g_{\alpha\beta}$. This gives:

$$\int\limits_{x'^0}^{x''^0} dx^0 \int\limits_{\Omega^{(3)}} \left[\frac{\partial\sqrt{-g}\,G}{\partial g_{\alpha\beta}} - \left(\frac{\partial\sqrt{-g}\,G}{\partial g_{\alpha\beta|\gamma}}\right)_{|\gamma}\right] \delta g_{\alpha\beta}\,d\mathbf{x} +$$

$$+ \int\limits_{x'^0}^{x''^0} dx^0 \int\limits_{\Sigma_{(2)}} \frac{\partial\sqrt{-g}\,G}{\partial g_{\alpha\beta|s}}\, n_s\, \delta g_{\alpha\beta}\, d\Sigma + \int\limits_{\Omega_{(3)}} \frac{\partial\sqrt{-g}\,G}{\partial g_{\alpha\beta|0}}\, \delta g_{\alpha\beta}\, d\mathbf{x}\,\Bigg|_{x'^0}^{x''^0}. \tag{4.1.19}$$

Here $\Sigma_{(2)}$ means a two-dimensional surface at infinity. Now the g's vary because we vary the ξ's and the $\xi_{|0}$'s in them. Therefore

$$\delta g_{\alpha\beta} = \sum_{B} g_{\alpha\beta|\overset{B}{\xi}{}^{s}} \delta \overset{B}{\xi}{}^{s} + \sum_{B} g_{\alpha\beta|\overset{B}{\xi}{}^{s}{}_{|0}} \delta \overset{B}{\xi}{}^{s}{}_{|0}. \tag{4.1.20}$$

Since we assume that the $\delta\xi$'s and their time derivatives vanish at the end of the time interval, we may set the last integral in (4.1.19) equal to zero. But it is different with the surface integral. Let us simply assume for the moment that the surface integral vanishes, that is we assume:

$$\int_{\Sigma_{(2)}} \frac{\partial\sqrt{-g}\,G}{\partial g_{\alpha\beta|s}} n_s \delta g_{\alpha\beta} d\sum = 0 \tag{4.1.21}$$

where $\delta g_{\alpha\beta}$ is (4.1.20). From the contents of Section 4, Chapter I follows the well-known formula:

$$\begin{aligned} \frac{\partial\sqrt{-g}\,G}{\partial g_{\alpha\beta}} - \left(\frac{\partial\sqrt{-g}\,G}{\partial g_{\alpha\beta|\gamma}}\right)_{|\gamma} &= -\left(\mathfrak{R}^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta}\mathfrak{R}\right) \\ &= 8\pi\mathfrak{T}^{\alpha\beta} = 8\pi \sum_{A} \overset{A}{\mu} \overset{A}{\xi}{}^{\alpha}{}_{|0} \overset{A}{\xi}{}^{\beta}{}_{|0} \overset{A}{\delta}. \end{aligned} \tag{4.1.22}$$

Therefore, taking into account our expression (4.1.20) for $\delta g_{\alpha\beta}$ we obtain:

$$\begin{aligned} \delta \int_{x'^0}^{x''^0} L''\,dx^0 = \sum_{A,B} 8\pi \{ &\overset{A}{\mu} \overset{A}{\overline{g_{\alpha\beta|\overset{B}{\xi}{}^{s}}}} \overset{A}{\xi}{}^{\alpha}{}_{|0} \overset{A}{\xi}{}^{\beta}{}_{|0} + \\ &- (\overset{A}{\mu} \overset{A}{\overline{g_{\alpha\beta|\overset{B}{\xi}{}^{s}{}_{|0}}}} \overset{A}{\xi}{}^{\alpha}{}_{|0} \overset{A}{\xi}{}^{\beta}{}_{|0})_{|0} \} \delta \overset{B}{\xi}{}^{s} + \sum_{A,B} \overset{A}{\mu} \overset{A}{\overline{g_{\alpha\beta|\overset{B}{\xi}{}^{s}{}_{|0}}}} \overset{A}{\xi}{}^{\alpha}{}_{|0} \overset{A}{\xi}{}^{\beta}{}_{|0} \delta \overset{B}{\xi}{}^{s} \Big|_{x'^0}^{x''^0}. \end{aligned} \tag{4.1.23}$$

Again the last expression vanishes because of the arbitrary $\delta\overset{B}{\xi}$. We have thus again obtained the wrong equations of motion, namely

$$\sum_{A} \overset{A}{\mu} \overset{A}{\overline{g_{\alpha\beta|\overset{B}{\xi}{}^{s}}}} \overset{A}{\xi}{}^{\alpha}{}_{|0} \overset{A}{\xi}{}^{\beta}{}_{|0} - \sum_{A} (\overset{A}{\mu} \overset{A}{\overline{g_{\alpha\beta|\overset{B}{\xi}{}^{s}{}_{|0}}}} \overset{A}{\xi}{}^{\alpha}{}_{|0} \overset{A}{\xi}{}^{\beta}{}_{|0})_{|0} = 0. \tag{4.1.24}$$

Our long argument is almost complete. We see that, if multiplied by $-\frac{1}{2}$, these are exactly the expressions that caused difficulty in (4.1.14). Thus, by a linear combination of these two Lagrangians we can obtain the right equations of motion provided that (4.1.21) is satisfied.

Therefore: we can obtain the equations of motion of the third kind from a Lagrangian:

$$L = -\sum_{A} \overset{A}{m}_{(0)} c(\overset{A}{\overline{g_{\alpha\beta}}}\overset{A}{\xi}{}^{\alpha}{}_{|0}\overset{A}{\xi}{}^{\beta}{}_{|0})^{1/2} + \frac{c^3}{16\pi k}\int_{\Omega(3)} \sqrt{-g}\,G\,d\mathbf{x} \tag{4.1.25}$$

where

$$G = g^{\mu\nu}\left(\begin{Bmatrix} \varrho \\ \nu\sigma \end{Bmatrix}\begin{Bmatrix} \sigma \\ \mu\varrho \end{Bmatrix} - \begin{Bmatrix} \varrho \\ \mu\nu \end{Bmatrix}\begin{Bmatrix} \sigma \\ \varrho\sigma \end{Bmatrix}\right). \tag{4.1.26}$$

In order that

$$\delta \int_{x'^0}^{x''^0} L\,dx^0 = 0 \tag{4.1.27}$$

may give us the right equations of motion, we assume that the variation of ξ^k and $\xi^k{}_{|0}$ vanishes at the end of the time interval and also that the equation

$$\int_{\Sigma(2)} d\sum \frac{\partial\sqrt{-g}G}{\partial g_{\alpha\beta|s}}\, n_s\,\delta g_{\alpha\beta} = 0 \tag{4.1.28}$$

is satisfied. We obtain the differential equations of the second order for the $3N$ quantities $\overset{A}{\xi}{}^k(x^0)$.

If we introduce the μ's we do it through

$$\overset{A}{\mu} = \frac{k\overset{A}{m}_{(0)}}{c^2}(\overset{A}{\overline{g_{\alpha\beta}}}\overset{A}{\xi}{}^{\alpha}{}_{|0}\overset{A}{\xi}{}^{\beta}{}_{|0})^{-1/2} \tag{4.1.29}$$

which we regard as the definition of the inertial mass appearing in the definition of $\mathcal{T}^{\alpha\beta}$:

$$\mathcal{T}^{\alpha\beta} = \sum_{A} \overset{A}{\mu}\overset{A}{\xi}{}^{\alpha}{}_{|0}\overset{A}{\xi}{}^{\beta}{}_{|0}\overset{A}{\delta}. \tag{4.1.30}$$

Thus, if only (4.1.28) is satisfied we can find the Lagrangian that we guessed in the previous chapter, using the definition (4.1.25) and checking, of course, whether (4.1.28) is satisfied. To this task we now turn.

2. THE LAGRANGIAN UP TO THE SIXTH ORDER

The calculation of the Lagrangian up to the sixth order is a very straightforward affair. But one amazing fact arises in it: we do not need to know $\underset{4}{h}_{00}$ in order to calculate such a Lagrangian. It is sufficient to know only $\underset{2}{h}_{00}$, $\underset{3}{h}_{0m}$ which, as we know, are very simple. Indeed, the only complication in finding the equations of motion of or up to the sixth order from the vanishing of $\mathcal{T}^{\alpha\beta}{}_{;\beta}$ consisted in finding $\underset{4}{h}_{00}$. This we do not need to do any longer. The only thing we must know about $\underset{4}{h}_{00}$ is that it must vanish at infinity like $1/r$. But (though we know it from Chapter III Section 2) we can find it without any explicit calculations of $\underset{4}{h}_{00}$. Indeed $\underset{4}{h}_{00|ss}$ can depend only on

$$\underset{2}{h}_{\underset{11}{00|00}} = \varphi_{|00};\ \varphi_{|s}\varphi_{|s};\ \varphi_{|ss}\varphi \tag{4.2.1}$$

But $\varphi_{|s}\varphi_{|s}$, $\varphi_{|ss}\varphi$ for $r \to \infty$ are of the order at most -4 in r. Therefore their contribution to $\underset{4}{h}_{00}$ is at most of order -2 in r. Now, about the contributions from $\varphi_{|00}$. For $r \to \infty$ we have for φ:

$$\varphi \sim \sum_{A} \frac{\overset{A}{m}}{r} - \sum_{A} \overset{A}{m}\, \overset{A}{\xi}{}^{s} \left(\frac{1}{r}\right)_{|s} + \dots \tag{4.2.2}$$

Because of the Newtonian equations of motion $\varphi_{|00}$ is at most of order -3, therefore the contributions coming from this expression are at most of order -1 in r. We shall very soon see that this fact insures the vanishing of the surface integral (4.1.28) which was a condition for the existence of such a Lagrangian. We may remark

in passing that $\underset{2}{h}_{00}$ and $\underset{3}{h}_{0m}$ are also of order -1 and all their derivatives are, because of the Newtonian equations of motion, at most of order -2 in r.

We shall start by calculating $\sqrt{-g}G$ up to the sixth order. A calculation which is straightforward, though slightly troublesome gives:

$$\underbrace{\sqrt{-g}G}_{4}+\underbrace{\sqrt{-g}G}_{6} = -\tfrac{1}{2}\varphi_{|m}\varphi_{|m}+\varphi\varphi_{|m}\varphi_{|m}-\tfrac{3}{2}\varphi_{|0}\varphi_{|0}+2\varphi_{|m}\underset{3}{h}_{0m|\underset{1}{0}}+$$

$$-\tfrac{1}{2}\underset{3}{h}_{0l|m}\underset{3}{h}_{0m|l}+\tfrac{1}{2}\underset{3}{h}_{0m|l}\underset{3}{h}_{0m|l}-\varphi_{|m}\underset{4}{h}_{00|m}. \tag{4.2.3}$$

We see that this expression is invariant with respect to the change

$$\underset{3}{h}_{0m}\to\underset{3}{h}_{0m}+a_{0|m},\qquad \underset{4}{h}_{00}\to\underset{4}{h}_{00}+2a_{0|0} \tag{4.2.4}$$

which is not astonishing in view of the fact that a_0, as we know, does not appear in the equations of motion. Therefore we may assume that $\underset{3}{h}_{0m}$ satisfies the coordinate condition

$$\underset{3}{\gamma}^{0s}{}_{|s}=\underset{3}{h}_{0s|s}=-\underset{2}{\gamma}^{00}{}_{|0}=2\underset{2}{h}_{00|0}. \tag{4.2.5}$$

We can rewrite (4.2.3) in the following form:

$$\underbrace{\sqrt{-g}G}_{4}+\underbrace{\sqrt{-g}G}_{6}=\tfrac{1}{2}\varphi\varphi_{|ss}-\tfrac{1}{2}\varphi^{2}\varphi_{|ss}+\tfrac{1}{8}\underset{3}{h}_{0l|l}\underset{3}{h}_{0m|m}+$$

$$-\tfrac{1}{2}\underset{3}{h}_{0a|ss}\underset{3}{h}_{0a}+\varphi_{|ss}\underset{4}{h}_{00}+W^{0}{}_{|0}+W^{m}{}_{|m}, \tag{4.2.6}$$

where W^0 and W^m are:

$$W^{0}=2\varphi_{|m}\underset{3}{h}_{0m},\quad W^{m}=-\tfrac{1}{2}(\varphi-\varphi^{2}+2\underset{4}{h}_{00})\varphi_{|m}-\tfrac{1}{2}(\underset{3}{h}_{0l}\underset{3}{h}_{0m})_{|l}+\tfrac{1}{2}\underset{3}{h}_{0l|m}\underset{3}{h}_{0l}. \tag{4.2.7}$$

From (4.2.3) we see that the function under the integral (4.1.28), the vanishing of which was a condition for the existence of our Lagrangian, is of order at most -3 in r. This is so since $\delta g_{\alpha\beta}$ is at most of order -1; therefore the integral vanishes; therefore the Lagrangian up to the sixth order exists.

Furthermore we see from the last two equations that forming the integral of $\sqrt{-g}G$, we can omit the last two expressions in (4.2.6). This is so for W^0 because the variation of ξ^k and $\xi^k{}_{|0}$ vanishes at the end of the time interval. This is so for W^m because W^m is at most of order -3 in r and therefore its surface integral vanishes. Thus, putting:

$$\varphi_{|ss} = 8\pi \sum_A \overset{A}{\mu} \overset{A}{\delta}, \qquad \underset{3}{h}_{0a|ss} = -16\pi \sum_A \overset{A}{\mu} \overset{A}{\dot{\xi}}{}^a{}_{|0} \overset{A}{\delta} \tag{4.2.8}$$

we can write:

$$\underset{4}{L''} + \underset{6}{L''} = \frac{c^3}{16\pi k} \int (\underbrace{\sqrt{-g}G}_{4} + \underbrace{\sqrt{-g}G}_{6})\, d\mathbf{x} = \frac{c^3}{k} \Big\{ \sum_A \overset{A}{\mu} [\tfrac{1}{4}\overset{A}{\tilde{\varphi}} - \tfrac{1}{4}\overset{A}{\tilde{\varphi}}{}^2 +$$

$$+ \tfrac{1}{2}\overset{A}{\tilde{h}}_{0a} \overset{A}{\xi}{}^a{}_{|0} + \tfrac{1}{2}\underset{4}{\overset{A}{\tilde{h}}}_{00}] + \frac{1}{8\cdot 16\pi} \int_{\Omega(3)} h_{0l|l} h_{0m|m}\, d\mathbf{x} \Big\}. \tag{4.2.9}$$

The last expression in the above equation requires special attention. We have:

$$h_{0a} = 4 \int d\mathbf{x}' \frac{\mathcal{T}'^{0a}}{|\mathbf{x}-\mathbf{x}'|}. \tag{4.2.10}$$

Therefore we can write:

$$\int h_{0l|l} h_{0m|m}\, d\mathbf{x} = 16 \int d\mathbf{x}'\, d\mathbf{x}''\, d\mathbf{x}\, \mathcal{T}'^{0l} \mathcal{T}''^{0m} \left(\frac{1}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}-\mathbf{x}''|} \right)_{|l'm''} \tag{4.2.11}$$

Let us first perform the integration

$$C = \int \frac{d\mathbf{x}}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}-\mathbf{x}''|}. \tag{4.2.12}$$

From this follows:

$$C_{|s's'} = -4\pi \int d\mathbf{x} \frac{\delta(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}''|} = -\frac{4\pi}{|\mathbf{x}'-\mathbf{x}''|}, \tag{4.2.13}$$

therefore

$$C = -2\pi|\mathbf{x}'-\mathbf{x}''|. \tag{4.2.14}$$

If we put this into (4.2.11) we can perform the other integrations simply as described in the previous chapter (Section 3) and we obtain:

$$\int h_{0l|l}h_{0m|m}\,d\mathbf{x} = -32\pi \sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}(r_{AB|\overset{A}{\xi}{}^{l}\overset{B}{\xi}{}^{m}})\, \overset{A}{\xi}{}^{l}{}_{|0}\, \overset{B}{\xi}{}^{m}{}_{|0}. \tag{4.2.15}$$

Therefore:

$$\underset{4}{L''}+\underset{6}{L''} = \frac{c^3}{k}\Big\{\sum_{A} \overset{A}{\mu}\,[\tfrac{1}{4}\overset{A}{\tilde{\varphi}}-\tfrac{1}{4}\overset{A}{\widetilde{\varphi^2}}+\tfrac{1}{2}\overset{A}{\tilde{h}}_{0a}\overset{A}{\xi}{}^{a}{}_{|0}+\tfrac{1}{2}\underset{4}{\overset{A}{\tilde{h}}}_{00}]+$$

$$-\tfrac{1}{4}\sum_{A,B}{}' \overset{A}{\mu}\overset{A}{\mu}\, r_{AB|\overset{A}{\xi}{}^{l}\overset{B}{\xi}{}^{m}}\, \overset{A}{\xi}{}^{l}{}_{|0}\, \overset{B}{\xi}{}^{m}{}_{|0}\Big\}. \tag{4.2.16}$$

Having finished with this part of the calculation, we now find

$$\underset{4}{L'}+\underset{6}{L'} = -\sum_{A} \overset{A}{m}_{(0)}\,c\,(\overset{A}{\tilde{g}}_{\alpha\beta}\,\overset{A}{\xi}{}^{\alpha}{}_{|0}\,\overset{B}{\xi}{}^{\beta}{}_{|0})^{1/2} \tag{4.2.17}$$

where $m_{(0)}$ is connected with μ, according to (4.1.13)

$$\overset{A}{m}_{(0)} = \frac{\overset{A}{\mu}c^2}{k}\,(\overset{A}{\tilde{g}}_{\alpha\beta}\overset{A}{\xi}{}^{\alpha}{}_{|0}\,\overset{A}{\xi}{}^{\beta}{}_{|0})^{1/2} \tag{4.2.18}$$

or

$$\overset{A}{m}_{(0)} = \frac{c^2}{k}\underset{2}{\overset{A}{\mu}} = \text{const.} \tag{4.2.19}$$

Therefore

$$\underset{4}{L'}+\underset{6}{L'} = -\frac{c^3}{k}\Big\{\sum_{A} \overset{A}{\mu}\big(1-\tfrac{1}{2}\overset{A}{\xi}{}^{s}{}_{|0}\,\overset{A}{\xi}{}^{s}{}_{|0}+\tfrac{1}{2}\overset{A}{\tilde{\varphi}}-\tfrac{1}{8}\overset{A}{\widetilde{\varphi^2}}-\tfrac{1}{8}(\overset{A}{\xi}{}^{a}{}_{|0}\,\overset{A}{\xi}{}^{a}{}_{|0})^2+$$

$$+\tfrac{3}{4}\overset{A}{\tilde{\varphi}}\overset{A}{\xi}{}^{a}{}_{|0}\,\overset{A}{\xi}{}^{a}{}_{|0}+\overset{A}{\tilde{h}}_{0a}\overset{A}{\xi}{}^{a}{}_{|0}+\tfrac{1}{2}\underset{4}{\overset{A}{\tilde{h}}}_{00}\big)\Big\}. \tag{4.2.20}$$

We see from (4.2.9) that the $\underset{4}{h}_{00}$ from L' and L'' cancel each other. Therefore we do not need to calculate $\underset{4}{h}_{00}$ explicitly. We also see from (4.2.20) that replacing

$$\underset{3}{h}_{0s} \text{ by } \underset{3}{h}_{0s}+a_{0|s} \text{ and } \underset{4}{h}_{00} \text{ by } \underset{4}{h}_{00}+2a_{0|0} \tag{4.2.21}$$

changes the Lagrangian by

$$\Delta L = -\frac{c^3}{k}\sum_A (\overset{A}{\overline{a}}_{0|s}\overset{A}{\xi}{}^s{}_{|0}+\overset{A}{\overline{a}}_{0|0}) = -\frac{c^3}{k}\Big(\sum_A \overset{A}{\overline{a}}_0\Big)_{|0} \tag{4.2.22}$$

which does not play any role in the variation of L and therefore in the equations of motion.

Omitting now the constant factor c^3/k, we obtain from (4.2.20) and (4.2.16):

$$L = -\sum_A \overset{A}{\mu}+\sum_A \big(\tfrac{1}{2}\overset{A}{\mu}\overset{A}{\xi}{}^a{}_{|0}\overset{A}{\xi}{}^a{}_{|0}+\tfrac{1}{8}\overset{A}{\mu}(\overset{A}{\xi}{}^a{}_{|0}\overset{A}{\xi}{}^a{}_{|0})^2\big)-\tfrac{1}{4}\sum_A \overset{A}{\mu}\overset{A}{\tilde{\varphi}}+$$
$$-\tfrac{1}{8}\sum_A \overset{A}{\mu}\overset{A}{\overline{\varphi^2}}-\tfrac{1}{2}\sum_A \overset{A}{\mu}\overset{A}{\overline{h}}_{0a}\overset{A}{\xi}{}^a{}_{|0}-\tfrac{3}{4}\sum_A \overset{A}{\mu}\overset{A}{\tilde{\varphi}}\overset{A}{\xi}{}^a{}_{|0}\overset{A}{\xi}{}^a{}_{|0}-\tfrac{1}{4}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}r_{AB|\overset{A}{\xi}{}^a\overset{B}{\xi}{}^b}\overset{A}{\xi}{}^a{}_{|0}\overset{B}{\xi}{}^b{}_{|0}. \tag{4.2.23}$$

Here we introduce in place of $\overset{A}{\tilde{\varphi}}$ and $\underset{3}{\overset{A}{\overline{h}}}_{0a}$:

$$\overset{A}{\tilde{\varphi}} = -2\sum_B \overset{B}{\mu}r_{AB}^{-1}, \qquad \underset{3}{\overset{A}{\tilde{h}}}_{0a} = 4\sum_B \overset{B}{\mu}r_{AB}^{-1}\overset{B}{\xi}{}^a{}_{|0} \tag{4.2.24}$$

and we finally obtain

$$L = -\sum_A \overset{A}{\mu}+\tfrac{1}{2}\sum_A \overset{A}{\mu}\overset{A}{\xi}{}^a{}_{|0}\overset{A}{\xi}{}^a{}_{|0}+\tfrac{1}{2}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}r_{AB}^{-1}+\tfrac{1}{8}\sum_A \overset{A}{\mu}(\overset{A}{\xi}{}^a{}_{|0}\overset{A}{\xi}{}^a{}_{|0})^2+$$
$$-\tfrac{1}{4}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}(\overset{A}{\mu}+\overset{B}{\mu})r_{AB}^{-2}-\tfrac{1}{6}\sum_{A,B,C}{}' \overset{A}{\mu}\overset{B}{\mu}\overset{C}{\mu}(r_{AB}^{-1}r_{BC}{}^{-1}+r_{BC}^{-1}r_{CA}^{-1}+$$
$$+r_{CA}^{-1}r_{AB}^{-1})-2\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}r_{AB}^{-1}\overset{B}{\xi}{}^a{}_{|0}\overset{A}{\xi}{}^a{}_{|0}+\tfrac{3}{4}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}(\overset{A}{\xi}{}^a{}_{|0}\overset{A}{\xi}{}^a{}_{|0}+$$
$$+\overset{B}{\xi}{}^a{}_{|0}\overset{B}{\xi}{}^a{}_{|0})r_{AB}^{-1}-\tfrac{1}{4}\sum_{A,B}{}' \overset{A}{\mu}\overset{B}{\mu}r_{AB|\overset{A}{\xi}{}^a\overset{B}{\xi}{}^b}\overset{A}{\xi}{}^a{}_{|0}\overset{B}{\xi}{}^b{}_{|0}. \tag{4.2.25}$$

This Lagrangian coincides (up to a trivial constant) with the one guessed in the previous chapter.

Thus we have not only found the Lagrangian of the Fokker type leading to the proper post-Newtonian equations of motion, but we have also formulated the general conditions under which such a Lagrangian exists. We saw that to calculate it we do not need to know $\underset{4}{h}_{00}$ explicitly. It was sufficient to know that $\underset{4}{h}_{00}$ vanishes like $1/r$ as $r \to \infty$.

Perhaps a historical remark would not be out of place. When the equations of motion were calculated with the help of certain surface integrals for the first time in 1938, we needed the knowledge of:

$$\underset{2}{h}_{00},\ \underset{2}{h}_{mn},\ \underset{3}{h}_{0n},\ \underset{4}{h}_{00},\ \underset{4}{h}_{mn},\ \underset{5}{h}_{0n}. \tag{4.2.26}$$

The first three were for the Newtonian approximation, the much more complicated last three for the post-Newtonian approximation. Later, we found that using the principle $\mathcal{T}^{\alpha\beta}{}_{;\beta} = 0$ we needed only

$$\underset{2}{h}_{00},\ \underset{2}{h}_{mn},\ \underset{3}{h}_{0n}\ ,\ \underset{4}{h}_{00}. \tag{4.2.27}$$

Now we find that to formulate the post-Newtonian equations of motion, we need as much knowledge of the metric field as we needed in 1938 for the Newtonian equations of motion. Or, today for the post-post-Newtonian equations of motion we need as much knowledge of the metric field as we needed before for post-Newtonian motion. Today we could calculate the post-post-Newtonian equations of motion with not too much difficulty if there were any physical sense in doing so.

3. GENERALIZATION

So far we have considered the motion of spherically symmetric non-rotating bodies represented by singularities of the single pole type. In this section, we shall show how to generalize the action

principle to include the case of rotating bodies which will be represented by the pole-dipole singularities of the gravitational field.

We saw from (4.1.22) that the variation with respect to $g_{\alpha\beta}$, if certain boundary conditions are fulfilled, gives the equation

$$\delta \int \sqrt{-g}\, G d\mathbf{x} = -\int (\mathfrak{R}^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta}\mathfrak{R})\, \delta g_{\alpha\beta} d\mathbf{x}. \tag{4.3.1}$$

Therefore the variational equation

$$\delta \int (\tfrac{1}{2}\sqrt{-g}\, G - 4\pi g_{\alpha\beta} \mathcal{T}^{\alpha\beta})\, d\mathbf{x} = 0 \tag{4.3.2}$$

(where $c = 1$ and the gravitational constant $k = 1$), when $\mathcal{T}^{\alpha\beta}$ does not depend on $g_{\mu\nu}$, gives the equation for the gravitational field.

Let us now use (4.3.2) as the definition of our Lagrange equation and vary its Lagrangian with respect to ξ^k; introducing as an example the case of single singularities — that is

$$\mathcal{T}^{\alpha\beta} = \sum_{A=1}^{N} \overset{A}{\mu} \overset{A}{\dot{\xi}^{\alpha}} \overset{A}{\dot{\xi}^{\beta}} \overset{A}{\delta}, \qquad \dot{\xi}^{\alpha} = \xi^{\alpha}{}_{|0} \tag{4.3.3}$$

then we do obtain the right equations of motion, identical with (1.7.8) if we assume μ as a known function of time. Therefore, in general we may regard the result of the variation of (4.3.2) with respect to ξ^k as giving us the right equations of motion, assuming of course that $\mathcal{T}^{\alpha\beta}$ does not depend on $g_{\mu\nu}$ and that μ is a function of time given by the added equation $\mathcal{T}^{0\beta}{}_{;\beta} = 0$. We may thus also expect to obtain correct equations of motion in the case of pole-dipole singularities representing rotating bodies and described by

$$\mathfrak{T}^{\alpha\beta} = \sum_{A=1}^{N} (\overset{A}{t}{}^{\alpha\beta} \overset{A}{\delta} - \overset{A}{t}{}^{r\alpha\beta} \overset{A}{\delta}{}_{|r}). \tag{4.3.4}$$

4. A FOKKER-TYPE LAGRANGIAN FOR ROTATING BODIES

Our aim is to obtain the Fokker-type Lagrangian leading to post-Newtonian equations of motion of rotating bodies. The bodies

are now characterized by a set of parameters $\overset{A}{t}{}^{\alpha\beta}$ and $\overset{A}{t}{}^{r\alpha\beta}$. According to the general considerations of Chapter III, the developments of these parameters for $(\alpha, \beta) = (0, 0)$, $(\alpha, \beta) = (0, r)$ and $(\alpha, \beta) = (r, s)$ should start with the orders 2, 3 and 4 respectively. But $\underset{2}{\overset{A}{t}}{}^{r00}$ would influence the Newtonian equations of motion through $\underset{2}{h}{}_{00}$. To avoid this complication, we shall start $\overset{A}{t}{}^{r00}$ with the third order.

The energy-momentum density (4.3.4) does not lead immediately to the correct Lagrangian. The coefficients $\overset{A}{t}{}^{\alpha\beta}$ and $\overset{A}{t}{}^{r\alpha\beta}$ must first be expressed in terms of the ξ^α and their time derivatives. This can be done by means of the equations

$$
\begin{aligned}
\mathfrak{T}^{\alpha\beta}{}_{;\beta} &= \mathfrak{T}^{\alpha0}{}_{|0} + \mathfrak{T}^{\alpha r}{}_{|r} + \mathfrak{T}^{00}\begin{Bmatrix}\alpha\\00\end{Bmatrix} + 2\mathfrak{T}^{0r}\begin{Bmatrix}\alpha\\0r\end{Bmatrix} + \mathfrak{T}^{rs}\begin{Bmatrix}\alpha\\rs\end{Bmatrix} \\
&= \sum_{A=1}^{N}\Bigg[\left(\overset{A}{t}{}^{\alpha0}{}_{|0} + \overset{A}{t}{}^{00}\begin{Bmatrix}\alpha\\00\end{Bmatrix} + 2\overset{A}{t}{}^{0r}\begin{Bmatrix}\alpha\\0r\end{Bmatrix} + \overset{A}{t}{}^{rs}\begin{Bmatrix}\alpha\\rs\end{Bmatrix}\right)\overset{A}{\delta} + \\
&\quad -\Bigg(\overset{A}{t}{}^{r\alpha0}{}_{|0} + \overset{A}{t}{}^{\alpha0}\overset{A}{\xi}{}^{r}{}_{|0} - \overset{A}{t}{}^{\alpha r} + \\
&\quad + \overset{A}{t}{}^{r00}\begin{Bmatrix}\alpha\\00\end{Bmatrix} + 2\overset{A}{t}{}^{rs0}\begin{Bmatrix}\alpha\\s0\end{Bmatrix} + \overset{A}{t}{}^{rst}\begin{Bmatrix}\alpha\\st\end{Bmatrix}\Bigg)\overset{A}{\delta}{}_{|r} + \\
&\quad + \left(\overset{A}{t}{}^{r\alpha0}\overset{A}{\xi}{}^{s}{}_{|0} - \overset{A}{t}{}^{r\alpha s}\right)\overset{A}{\delta}{}_{|rs}\Bigg],
\end{aligned}
\tag{4.4.1}
$$

which by integration over the three-dimensional neighbourhoods of the singularities lead to

$$
\begin{aligned}
\int_{\overset{A}{\Omega}} \mathfrak{T}^{\alpha\beta}{}_{;\beta}\,d\mathbf{x} &= \overset{A}{t}{}^{\alpha0}{}_{|0} + \overset{A}{t}{}^{00}\overset{A}{\overline{\begin{Bmatrix}\alpha\\00\end{Bmatrix}}} + 2\overset{A}{t}{}^{0r}\overset{A}{\overline{\begin{Bmatrix}\alpha\\0r\end{Bmatrix}}} + \\
&\quad + \overset{A}{t}{}^{rs}\overset{A}{\overline{\begin{Bmatrix}\alpha\\rs\end{Bmatrix}}} + \overset{A}{t}{}^{r00}\overset{A}{\overline{\begin{Bmatrix}\alpha\\00\end{Bmatrix}_{|r}}} + 2\overset{A}{t}{}^{rs0}\overset{A}{\overline{\begin{Bmatrix}\alpha\\s0\end{Bmatrix}_{|r}}} + \overset{A}{t}{}^{rst}\overset{A}{\overline{\begin{Bmatrix}\alpha\\st\end{Bmatrix}_{|r}}} = 0,
\end{aligned}
\tag{4.4.2}
$$

$$\int\limits_{\overset{A}{\Omega}} (x^r - \overset{A}{\xi}{}^r)\mathfrak{T}^{a\beta}{}_{;\beta} d\mathbf{x} = \overset{A}{t}{}^{ra0}{}_{|0} + \overset{A}{t}{}^{a0}\overset{A}{\xi}{}^r{}_{|0} - \overset{A}{t}{}^{ar} + \overset{A}{t}{}^{r00}\overbrace{\begin{Bmatrix} a \\ 00 \end{Bmatrix}}^{A} +$$

$$+2\overset{A}{t}{}^{rs0}\overbrace{\begin{Bmatrix} a \\ s0 \end{Bmatrix}}^{A} + \overset{A}{t}{}^{rst}\overbrace{\begin{Bmatrix} a \\ st \end{Bmatrix}}^{A} = 0, \qquad (4.4.3)$$

$$\int\limits_{\overset{A}{\Omega}} (x^r - \overset{A}{\xi}{}^r)(x^s - \overset{A}{\xi}{}^s)\mathfrak{T}^{a\beta}{}_{;\beta} d\mathbf{x} = \overset{A}{t}{}^{ra0}\overset{A}{\xi}{}^s{}_{|0} + \overset{A}{t}{}^{sa0}\overset{A}{\xi}{}^r{}_{|0} - \overset{A}{t}{}^{ras} - \overset{A}{t}{}^{sar} = 0. \quad (4.4.4)$$

Equations (4.4.2) for $a = 0$ give in the third order:

$$\underset{3}{\overset{A}{t}}{}^{00}{}_{|0} = \underset{3}{\overset{A}{\mu}}{}_{|0} = 0. \qquad (4.4.5)$$

Similarly (4.4.3) give

$$\underset{3}{\overset{A}{t}}{}^{0r} = \underset{2}{\overset{A}{\mu}}\overset{A}{\xi}{}^r{}_{|0}, \qquad (4.4.6)$$

and (4.4.4) give

$$\underset{3}{\overset{A}{t}}{}^{r0s} + \underset{3}{\overset{A}{t}}{}^{s0r} = 0. \qquad (4.4.7)$$

Introducing the notation:

$$\underset{3}{\overset{A}{t}}{}^{r0s} - \underset{3}{\overset{A}{t}}{}^{s0r} = \underset{3}{\overset{A}{S}}{}^{rs} \qquad (4.4.8)$$

from (4.4.7) we have

$$\underset{3}{\overset{A}{t}}{}^{r0s} = -\underset{3}{\overset{A}{t}}{}^{s0r} = \tfrac{1}{2}\underset{3}{\overset{A}{S}}{}^{rs}. \qquad (4.4.9)$$

Here $\underset{3}{\overset{A}{S}}{}^{rs}$ should be interpreted as the Newtonian internal angular momentum of the A'th body since it is equal to

$$\int\limits_{\overset{A}{\Omega}} [(x^r - \overset{A}{\xi}{}^r)\underset{3}{\mathfrak{T}}{}^{0s} - (x^s - \overset{A}{\xi}{}^s)\underset{3}{\mathfrak{T}}{}^{0r}]\,d\mathbf{x}. \tag{4.4.10}$$

In the fourth order and for $\alpha = m$ equations (4.4.2) give the Newtonian equations of motion. In the fourth order and for $\alpha = m$ equations (4.4.3) and (4.4.4) give:

$$\underset{4}{\overset{A}{t}}{}^{mr} = \underset{2}{\overset{A}{\mu}}\,\overset{A}{\xi}{}^m{}_{|0}\,\overset{A}{\xi}{}^r{}_{|0}, \tag{4.4.11}$$

$$\underset{3}{\overset{A}{S}}{}^{mr}{}_{\underset{1}{|0}} = 0 \tag{4.4.12}$$

and

$$\underset{4}{\overset{A}{t}}{}^{rst} = \tfrac{1}{2}(\underset{3}{\overset{A}{S}}{}^{rs}\,\overset{A}{\xi}{}^t{}_{|0} + \underset{3}{\overset{A}{S}}{}^{rt}\,\overset{A}{\xi}{}^s{}_{|0}). \tag{4.4.13}$$

Equations (4.4.12) mean that the Newtonian internal angular momenta are conserved, as should be expected. We may summarize the results obtained thus far as follows: the values of the $\overset{A}{t}{}^{\alpha\beta}$ in the approximation considered are unaffected by the rotation and are the same as for non-rotating bodies; the quantities $\overset{A}{t}{}^{r0s}$ and $\overset{A}{t}{}^{rst}$ are expressed in terms of constant parameters $\overset{A}{S}{}^{rs}$ representing rotation.

Let us go on to the next approximation. In the fifth and sixth order equations (4.4.2) define

$$\underbrace{(\overset{A}{t}{}^{00}\,\overset{A}{\xi}{}^m{}_{|0})}_{5}{}_{\underset{1}{|0}} - \underset{5}{\overset{A}{t}}{}^{0m}{}_{\underset{1}{|0}} = \underbrace{(\overset{A}{\mu}\,\overset{A}{\xi}{}^m{}_{|0})}_{6}{}_{|0} - \underset{6}{\overset{A}{t}}{}^{0m}{}_{|0} \tag{4.4.14}$$

through the terms of the order 2, 3 and 4. But the expression (4.4.14) would be exactly zero only in the Newtonian case in which $t^{0m} = t^{00}\xi^m{}_{|0} = \mu\xi^m{}_{|0}$. However, in the case of rotating bodies,

this equals a small quantity:

$$\underset{6}{\overset{A}{q}}{}^{0m} = \underbrace{(\overset{A}{t}{}^{00}\overset{A}{\xi}{}^{m}{}_{|0})_{|0}}_{6} - \underbrace{\overset{A}{t}{}^{0m}{}_{|0}}_{6} \tag{4.4.15}$$

which we can calculate explicitly from (4.4.2), though this would be quite troublesome. Going further, for $\alpha = 0$ we can write the equation (4.4.3), after differentiating it with respect to time, in the form

$$\underbrace{\overset{A}{t}{}^{r00}{}_{|00}}_{6} + \underset{6}{\overset{A}{q}}{}^{0r} + 2\underset{3}{\overset{A}{t}}{}^{rs0}\underset{2}{\overset{A}{\overline{\begin{Bmatrix}0\\ s0\end{Bmatrix}}}}{}_{\underset{1}{|0}} = 0, \tag{4.4.16}$$

which gives

$$\underset{4}{\overset{A}{t}}{}^{r00} = \underset{3}{\overset{A}{S}}{}^{rs}\dot{\overset{A}{\xi}}{}^{s} + \int_{t_0}^{t}\Big(\underset{5}{\overset{A}{t}}{}^{0r} - \underbrace{\overset{A}{t}{}^{00}\overset{A}{\xi}{}^{r}{}_{|0}}_{5}\Big)dt. \tag{4.4.17}$$

Therefore, $\underset{4}{t}{}^{r00}$ is determined by equations (4.4.2) and (4.4.3.) when $\alpha = 0$. Then we may use equations (4.4.3) for $\alpha = m$ as determining $\underset{6}{t}{}^{mr}$ if $\underset{5}{t}{}^{rm0}$ is arbitrary. Equation (4.4.4) determines $\underset{6}{t}{}^{rms}$, and so on. We also observe that condition (4.1.21) is satisfied since additional terms in the gravitational field are of the order r^{-2}. The additional term in $\underset{3}{h}_{0r}$ is

$$\underset{3}{h'}_{0r} = -2\sum_{A=1}^{N}{}' \overset{A}{S}{}^{sr}(\overset{A}{r}{}^{-1})_{|s}. \tag{4.4.18}$$

It follows that the post-Newtonian Lagrangian for rotating bodies does exist and has the form

$$\int\left(-\tfrac{1}{2}\mathfrak{T}^{\alpha\beta}g_{\alpha\beta} + \frac{1}{16\pi}\sqrt{-g}\;G\right)d\mathbf{x}. \tag{4.4.19}$$

In this Lagrangian we are looking for new terms connected with the rotation. These are:

$$-\sum_{A=1}^{N}\Big(\underset{3}{\overset{A}{t}}{}^{0r}\,\overline{\underset{3}{h'}_{0r}}^{A}+\tfrac{1}{2}\underset{4}{\overset{A}{t}}{}^{r00}\,\overline{\underset{2}{h}_{00|r}}^{A}+\underset{3}{\overset{A}{t}}{}^{r0s}\,\overline{\underset{3}{h}_{0s|r}}^{A}+\tfrac{1}{2}\underset{4}{\overset{A}{t}}{}^{rst}\,\overline{\underset{2}{h}_{st|r}}^{A}\Big)+$$

$$-\frac{1}{32\pi}\int \underset{3}{h}_{0r|ss}\underset{3}{h}_{0r}\,d\mathbf{x}$$

$$=-\tfrac{1}{2}\sum_{A=1}^{N}\Big(\underset{3}{\overset{A}{t}}{}^{0r}\,\overline{\underset{3}{h'}_{0r}}^{A}+\underset{4}{\overset{A}{t}}{}^{r00}\,\overline{\underset{2}{h}_{00|r}}^{A}+\underset{3}{\overset{A}{t}}{}^{r0s}\,\overline{\underset{3}{h}_{0s|r}}^{A}+\underset{4}{\overset{A}{t}}{}^{rst}\,\overline{\underset{2}{h}_{st|r}}^{A}\Big). \tag{4.4.20}$$

Neglecting terms proportional to the squares of angular momenta we finally obtain, for a system of two rotating bodies:

$$2\overset{1}{\mu}\overset{2}{S}{}^{sr}\,\overset{1}{\xi}{}^{r}{}_{|0}\left(\frac{1}{r}\right)_{|\overset{1}{\xi}{}^{s}}-2\overset{1}{\mu}\overset{2}{S}{}^{sr}\,\overset{2}{\xi}{}^{r}{}_{|0}\left(\frac{1}{r}\right)_{|\overset{1}{\xi}{}^{s}}+$$

$$-2\overset{2}{\mu}\overset{1}{S}{}^{sr}\,\overset{2}{\xi}{}^{r}{}_{|0}\left(\frac{1}{r}\right)_{|\overset{1}{\xi}{}^{s}}+2\overset{2}{\mu}\overset{1}{S}{}^{sr}\,\overset{1}{\xi}{}^{r}{}_{|0}\left(\frac{1}{r}\right)_{|\overset{1}{\xi}{}^{s}}+$$

$$-\overset{1}{\mu}\left(\frac{1}{r}\right)_{|\overset{1}{\xi}{}^{r}}\int_{t_0}^{t}\Big(\underset{5}{\overset{2}{t}}{}^{0r}-\underbrace{\overset{2}{t}{}^{00}\,\overset{2}{\xi}{}^{r}{}_{|0}}_{5}\Big)dt+\overset{2}{\mu}\left(\frac{1}{r}\right)_{|\overset{1}{\xi}{}^{r}}\int_{t_0}^{t}\Big(\underset{5}{\overset{1}{t}}{}^{0r}-\underbrace{\overset{1}{t}{}^{00}\,\overset{1}{\xi}{}^{r}{}_{|0}}_{5}\Big)dt \tag{4.4.21}$$

which we shall consider in the next chapter.

CHAPTER V

THE ONE AND TWO PARTICLE PROBLEMS

1. ON THE QUESTION OF MEASUREMENT

We shall specialize our Lagrangian to the case of one particle, then to two particles and we shall find the motion in the chosen coordinate system, characterized by the approximation method. But before doing so we must know the physical meaning of our calculations. As we well know, the one-body problem — that is that of a small planet travelling around the sun — can be solved rigorously without too great difficulty. The result is that the planet moves along an ellipse, as in Newtonian theory, and that the ellipse very slowly rotates. But what do these statements mean? If they refer to a special coordinate system, then there is little physical meaning in such statements if we do not characterize the coordinate system within which they are valid. Yet, if we have to characterize the special coordinate system, then we violate the spirit of General Relativity Theory; its statements should be valid in any (or almost any) coordinate system. Some authors consider that the only way out of this dilemma is to introduce certain definite coordinate systems. This seems to us (as we mentioned in Chapter I) essentially a step backwards from the ideas introduced by G. R. T.

Yet there is a simple way out of these difficulties. We have merely to admit coordinate systems which are Galilean at infinity. That is, coordinate systems in which for $r \to \infty$ the metric form is

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta. \tag{5.1.1}$$

There, we can speak about ideal rigid rods and about ideal clocks.

But physical events do not happen there. They happen in the Riemannian space with arbitrary coordinate systems satisfying only the condition that they go over into Galilean coordinate systems at infinity. The Galilean system is inertial and, of course, there exists a family of such systems whose members are connected by Lorentz transformations. However, we shall speak only about one member of the family. In the case of the sun, it will be the Galilean coordinate system in which the sun is at rest; in the case of a two-body system, it will be the coordinate system in which the center of gravity defined by (3.3.4) or one of the two bodies is at rest.

How can we connect the events happening in our arbitrary coordinate system with an event in the Galilean coordinate system? Let us call τ and ξ^k the time and space coordinates of a particular event. An observer in the Galilean coordinate system learns about the event through a light-ray sent to him and reaching him at the point Ξ^k and at the moment T. In this way, every event in the Riemannian space is projected into the Galilean coordinate system very far from the event, where the influence of the gravitational field can be neglected. Let us assume that there are a number of events taking place around the sun at rest, all of them lying in a plane. (For the moment we omit the difficulty connected with the meaning of defining events lying in one plane.) Let us call this plane the (x, y) plane. From each event a light-ray signal is sent so that it reaches the (X, Y) plane in a Galilean coordinate system. The light-rays are so sent that they all fall perpendicular to the (X, Y) plane which is parallel to the (x, y) plane. (Again we omit the difficulty connected with the meaning of parallel planes.) The description of the pattern that the τ, ξ^1, ξ^2 events form on the (x, y) plane depends on the choice of the coordinate system. The pattern of the corresponding signals T, Ξ^1, Ξ^2 on the Galilean (X, Y) plane permits an objective description, since we know the construction of the rods and clocks by which the times and positions of these events can be ascertained. Thus our task is to find the correspondence between the τ, ξ^k and the T, Ξ^k events, in order to be able to pass from the subjective to the objective description of the events. If we change the coordinate systems internally, keeping the Galilean

unchanged, then τ, ξ^k changes, but T, Ξ^k does not. To give a mechanical example, imagine a rigid plane from which wires go out, ending in a rubber plane. We may change and deform the rubber plane, change the position of the wires, as long as we keep the upper plane firm and the ending of the wires in it. The comparison with our example is obvious: the rigid upper plane stands for our Galilean coordinate system. The rubber is our coordinate system in the Riemannian manifold; the wires are the light rays.

We shall therefore start our mathematical analysis by finding the connection between the τ, ξ^k event and the corresponding T, Ξ^k event. Since all test particles, independent of their mass, move along the geodesic, so also does the light-ray. This is partially an assumption, but a well-founded one. It is a null geodesic along which the light-ray moves. Since $ds = 0$ and there is no place for the concept of mass in it, it will be best to adopt a form of the equations of motion in which ds does not appear; a form in which the time t is the parameter. For the sake of brevity from now on we shall assume:

$$c = 1, \quad k = 1, \quad x^0 = t, \quad \dot{x}^k = \frac{dx^k}{dt} = x^k{}_{|0}.$$

The equation of a geodesic line with t as a parameter is:

$$\frac{d}{dt}(\lambda \dot{x}^k) + \lambda \left\{ {k \atop \alpha\beta} \right\} \dot{x}^\alpha \dot{x}^\beta = 0. \tag{5.1.2}$$

These are three equations which will allow us to find the three x's; the fourth one which determines λ is:

$$g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0 \tag{5.1.3}$$

which states that the geodesic line is a null line.

Let us start by completely neglecting the gravitational field. Then we have:

$$\ddot{x}^k + \frac{\dot{\lambda}}{\lambda} \dot{x}^k = 0, \qquad \dot{x}^k \dot{x}^k = 1. \tag{5.1.4}$$

From this follows:

$$\frac{\dot{\lambda}}{\lambda} = 0, \qquad x^k = \xi^k + n^k(t-\tau), \qquad n^k n^k = 1. \tag{5.1.5}$$

This is the world-line of a light-ray that was emitted from a point ξ^k at the moment τ in the n^k direction, in a Euclidean space and in a Galilean coordinate system. Indeed, we have:

$$\dot{x}^k = n^k, \qquad \ddot{x}^k = 0, \qquad x^k = \xi^k \quad \text{for} \quad t = \tau. \tag{5.1.6}$$

Now we wish to take into account the gravitational field. Obviously our approximation procedure, based on the smallness of the velocity of the moving bodies compared to the velocity of light is successful only if such a velocity is smaller than that of light. Obviously we cannot use this method for light-rays. Therefore, for our next approximation we shall assume: if x^k is the point on the world-line of the light-ray then its order is not changed by taking its time derivative. Therefore for the world-line of the light-ray we have:

$$\left.\begin{aligned} x^k &= \underset{0}{x}^k + \underset{1}{x}^k + \underset{2}{x}^k + \ldots, \\ \dot{x}^k &= \underset{0}{\dot{x}}^k + \underset{1}{\dot{x}}^k + \underset{2}{\dot{x}}^k + \ldots, \\ \ddot{x}^k &= \underset{0}{\ddot{x}}^k + \underset{1}{\ddot{x}}^k + \underset{2}{\ddot{x}}^k + \ldots. \end{aligned}\right\} \tag{5.1.7}$$

All our other assumptions remain unchanged. This means: the differentiation with respect to time of all other quantities increases the order by one. Therefore, in the next approximation, since

$$\underset{2}{h}_{00} = g_{00} - 1 = \varphi, \qquad \underset{2}{h}_{mn} = g_{mn} + \delta_{mn} = \delta_{mn}\varphi \tag{5.1.8}$$

we have from (5.1.2) and (5.1.3):

$$\underset{2}{\ddot{x}}^k + (\log \underset{2}{\lambda})_{|0}\,\underset{0}{\dot{x}}^k = \varphi_{|m}\,\underset{0}{\dot{x}}^m\,\underset{0}{\dot{x}}^k - \varphi_{|k}, \tag{5.1.9a}$$

$$\underset{2}{\dot{x}}^k\,\underset{0}{\dot{x}}^k = \varphi. \tag{5.1.9b}$$

From this we see that $\underset{1}{x}^k = 0$, and equations (5.1.9) allow us to find the world-line of the light-ray in the next approximation.

We shall assume, without much loss of generality, that we are dealing only with a one-body problem, that is:

$$\varphi = -\frac{2\mu}{r}, \quad r^2 = x^k x^k. \tag{5.1.10}$$

The extension of the calculation to the many-body problem would cause very little change, since in this approximation the theory is a linear one and the motion of the bodies would affect only the light-rays in the next approximation, in which we are not interested.

We regard as our zero approximation that referring to the absence of the gravitational field, that is the one in (5.1.5).

Multiplying (5.1.9a) by $\underset{0}{\dot{x}}^k$, we obtain:

$$(\log \lambda)_{|0} = -\dot{\varphi} = -\varphi_{|s} \underset{0}{\dot{x}}^s, \tag{5.1.11}$$

and because of (5.1.9a) the equation for $\underset{2}{x}^k$ is:

$$\underset{2}{\ddot{x}}^k = 2\varphi_{|m} \underset{0}{n}^m \underset{0}{n}^k - \varphi_{|k}. \tag{5.1.12}$$

We shall now make a special assumption about the vector $\underset{0}{n}^k$. We shall assume

$$\xi^s \underset{0}{n}^s = 0 \tag{5.1.13}$$

that is, that in the zeroth approximation the light-ray is sent perpendicular to the vector pointing from the sun to the point from which the light-ray is sent. We make this assumption for two reasons: firstly, from the point of view of observation this is the only case in which we are really interested; secondly, although the general case in which (5.1.13) is not assumed could be solved almost as easily, still the formulas are much more clumsy.

Thus our simple problem now is to solve (5.1.12), assuming (5.1.13). Instead of (5.1.12) we write, because of (5.1.10):

$$\ddot{\underset{2}{x}}^k = \frac{4\mu}{r^3} x^s \underset{0}{n}^s \underset{0}{n}^k - \frac{2\mu}{r^3} x^k. \tag{5.1.14}$$

Since μ is of the order two, we can introduce into this equation the expressions (5.1.5) for x^k that is, the Galilean solutions. We shall now write, because of (5.1.13):

$$r^2 = a^2 + (t-\tau)^2, \qquad \xi^s \xi^s = a^2 \tag{5.1.15}$$

and for (5.1.14) we have a simple differential equation:

$$\ddot{\underset{2}{x}}^k = \frac{2\mu}{\left(a^2 + (t-\tau)^2\right)^{3/2}} \left(\underset{0}{n}^k (t-\tau) - \underset{0}{\xi}^k\right). \tag{5.1.16}$$

By integration of the last equation we find:

$$\dot{\underset{2}{x}}^k = \underset{2}{n}^k = -\frac{2\mu}{\left(a^2 + (t-\tau)^2\right)^{1/2}} \left(\underset{0}{n}^k + \frac{\xi^k (t-\tau)}{a^2}\right) + \underset{2}{b}_k \tag{5.1.17}$$

where $\underset{2}{b}_k$ is the arbitrary constant to be determined by some added conditions. As such we wish to assume:

$$\underset{2}{n}^k = 0 \quad \text{for} \quad t = \infty. \tag{5.1.18}$$

This means: far from the gravitational field the light-ray has the prescribed direction, which is perpendicular to ξ^k. That is, for $t = \infty$:

$$\dot{\underset{2}{x}}^k \big|_{t=\infty} = -\frac{2\mu \xi^k}{a^2} + \underset{2}{b}_k = 0, \tag{5.1.19}$$

therefore

$$\underset{2}{n}^k = \dot{\underset{2}{x}}^k = -\frac{2\mu}{\left(a^2 + (t-\tau)^2\right)^{1/2}} \left(\underset{0}{n}^k + \frac{\xi^k (t-\tau)}{a^2}\right) + \frac{2\mu \xi^k}{a^2}. \tag{5.1.20}$$

For $t = \tau$ we have:

$$\underset{2}{n}^k \big|_{t=\tau} = \frac{2\mu}{a^2} \left(\xi^k - \underset{0}{n}^k a\right). \tag{5.1.21}$$

Therefore $\underset{0}{n}^k+\underset{2}{n}^k$ must have a small component in the ξ^k direction at the time $t=\tau$, in order to be perpendicular to the ξ^k direction at time $t=\infty$. This is the famous formula for the deflection of light-rays. For $t=-\infty$ it follows from (5.1.20) that the component in the ξ^k direction would be

$$\xi^k \underset{2}{n}^k|_{t=-\infty} = 4\mu. \tag{5.1.22}$$

Now we wish to integrate (5.1.17) again, under the condition

$$\underset{2}{x}^k|_{t=\tau} = 0 \tag{5.1.23}$$

because ξ^k is the exact point on the light-ray for $t=\tau$. We obtain:

$$\begin{gathered} \underset{2}{x}^k = -2\mu\underset{0}{n}^k\log\frac{r+t-\tau}{a}-\frac{2\mu r\xi^k}{a^2}+\frac{2\mu\xi^k}{a}\left(1+\frac{t-\tau}{a}\right), \\ r^2 = \xi^s\xi^s+(t-\tau)^2 = a^2+(t-\tau)^2. \end{gathered} \tag{5.1.24a}$$

We see that this integral of (5.1.20) does indeed satisfy the conditions (5.1.23). Now for $t\to\infty$ we have:

$$\underset{2}{x}^k(t) \approx \frac{2\mu\xi^k}{a}-2\mu\underset{0}{n}^k\log\frac{2(t-\tau)}{a}. \tag{5.1.24b}$$

On the right-hand side of this formula we have omitted all expressions which go to 0 as $t\to\infty$. We are essentially interested in the projection of the events ξ^k, τ, into the far-away Galilean system, that is for $t\to\infty$. Let us now, as in our introductory remarks to this section, call

$$\Xi^k = \underset{0}{x}^k+\underset{2}{x}^k \quad \text{for} \quad t\to\infty. \tag{5.1.25}$$

Then, because of (5.1.24b) and (5.1.5), we have:

$$\Xi^k = \xi^k\left(1+\frac{2\mu}{a}\right)+\underset{0}{n}^k(t-\tau)-2\mu\underset{0}{n}^k\log\frac{2(t-\tau)}{a} \tag{5.1.26}$$

and

$$\dot{\Xi}^k = \underset{0}{n}^k. \tag{5.1.27}$$

The formulas (5.1.26) and (5.1.27) are valid with the same accuracy as (5.1.24b). They are the result of our calculations connecting the events in our coordinate (Newtonian) system defined by our approximation procedure with the events in the far-removed Galilean coordinate system. But we should remember that these formulas were obtained only under the assumption that $\underset{0}{n}^k$ is perpendicular to ξ^k. We shall use these formulas when we try to interpret the motion of a particle in an objective way. In any case we see that, generally, because μ/a is small, the approximation

$$\Xi^k = \xi^k + \underset{0}{n}^k(t-\tau) \tag{5.1.28}$$

is a good one. Or, with a great degree of accuracy, we may interpret the events in our Riemannian coordinate system in a Galilean way, as long as they are not too near the sun. For example, if we consider only the earth and the sun with mass M then for an event on the earth we have:

$$a \approx 1.5 \cdot 10^8\,\mathrm{km}, \qquad \mu = \frac{kM}{c^2} \approx 1.5\,\mathrm{km}. \tag{5.1.29}$$

Therefore

$$\frac{2\mu}{a} \approx 2 \cdot 10^{-8}. \tag{5.1.30}$$

As an example, let us consider an event for which

$$\tau = 0, \qquad \xi^1 = a, \qquad \xi^2 = \xi^3 = 0, \tag{5.1.31}$$

and since $\xi^s \underset{0}{n}^s = 0$:

$$\underset{0}{n}^1 = \underset{0}{n}^2 = 0, \qquad \underset{0}{n}^3 = 1. \tag{5.1.32}$$

Then we have (compare fig. 1):

$$\Xi^1 = a+2\mu, \qquad \Xi^2 = 0, \qquad \Xi^3 = t\left(1-\frac{2\mu}{t}\log\frac{t}{a}\right). \tag{5.1.33}$$

Or, for a sufficiently great t:

$$\Xi^1 = a+2\mu, \qquad \Xi^2 = 0, \qquad \Xi^3 = t. \tag{5.1.34}$$

We shall return to the contents of this section in order to interpret the motion of particles objectively.

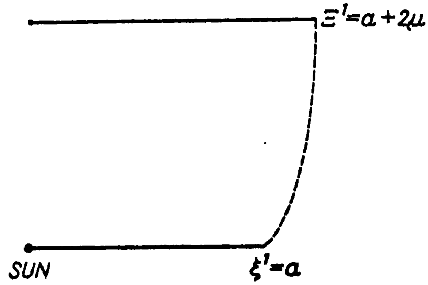

Fig. 1.

Connection between a subjective and objective representation of an event

2. ON THE MOTION OF A TEST PARTICLE IN THE FIELD OF A HEAVY PARTICLE

We know that the problem of a test particle moving in the field of the sun can be solved rigorously; however, using our general formulas here, we shall find the motion in the post-Newtonian approximation without making use of the explicit metric form or the equation of the geodesic line.

We start from the Lagrangian and assume:

$$\overset{1}{\xi}{}^k \equiv 0, \qquad \overset{2}{\xi}{}^k = \xi^k, \qquad \overset{1}{\mu} = \mu, \qquad \overset{2}{\mu} = \Delta\mu. \tag{5.2.1}$$

We shall consistently neglect all products of $\Delta\mu$. Then, dividing the old Lagrangian by $\Delta\mu$, for the new Lagrangian L we have:

$$\underset{2}{L}+\underset{4}{L} = \tfrac{1}{2}\xi^a{}_{|0}\,\xi^a{}_{|0} + \frac{\mu}{r} + \tfrac{1}{8}(\xi^a{}_{|0}\xi^a{}_{|0})^2 + \frac{3}{2}\frac{\mu}{r}\,\xi^a{}_{|0}\,\xi^a{}_{|0} - \tfrac{1}{2}\frac{\mu^2}{r^2}. \tag{5.2.2}$$

It is now more convenient to introduce instead of x^0 a new variable

$$dx^0 = \left(1+\frac{4\mu}{r}\right)dx'^0 \tag{5.2.3}$$

so that

$$L dx^0 = L' dx'^0. \tag{5.2.4}$$

Denoting the derivatives with respect to x'^0 by $0'$, we have:

$$\underset{2}{L'} + \underset{4}{L'} = \frac{1}{2}\,\xi^a{}_{|0'}\xi^a{}_{|0'} + \frac{\mu}{r} + \frac{1}{2}\left(\frac{1}{2}\,\xi^a{}_{|0'}\xi^a{}_{|0'} - \frac{\mu}{r}\right)^2 + \frac{3\mu^2}{r^2}. \tag{5.2.5}$$

The expression in the bracket in the above equation is constant because of the Newtonian equations of motion. Ignoring it, we obtain:

$$\underset{2}{L'} + \underset{4}{L'} = \tfrac{1}{2}\xi^a{}_{|0'}\xi^a{}_{|0'} + \frac{\mu}{r} + \frac{3\mu^2}{r^2}. \tag{5.2.6}$$

This Lagrangian corresponds to that in classical mechanics with the potential function

$$V = -\frac{\mu}{r} - \frac{3\mu^2}{r^2}. \tag{5.2.7}$$

We have as its integrals the law of conservation of momenta, which in the polar coordinate system (φ, r) are:

$$r^2\varphi_{|0'} = \underset{1}{J} + \underset{3}{J} \tag{5.2.8}$$

and the law of conservation of energy:

$$\tfrac{1}{2}[(r_{|0'})^2 + r^2(\varphi_{|0'})^2] - \frac{\mu}{r} - \frac{3\mu^2}{r^2} = \underset{2}{E} + \underset{4}{E}. \tag{5.2.9}$$

The problem is therefore reduced to the one well-known in classical mechanics. Its solution, written in a parametric form, is:

$$r = a(1 - \varepsilon\cos u), \tag{5.2.10a}$$

$$\varphi = \left(1 + \frac{3\mu}{(1-\varepsilon^2)a}\right)\cdot 2\arctan\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{u}{2}, \tag{5.2.10b}$$

$$x'^0 = \sqrt{\frac{a^3}{\mu}}\left(1 + \frac{\mu}{2a}\right)(u - \varepsilon\sin u). \tag{5.2.10c}$$

The constants a and ε are determined by the constants E and J. However, it is more convenient to regard a and ε instead of E and J as given in advance. That (5.2.10) are solutions of the equations (5.2.8), (5.2.9) can be checked by substitution. In the case $\varepsilon > 1$, we have to replace

$$a \text{ by } -a \quad \text{and} \quad u \text{ by } -iu. \tag{5.2.11}$$

We now wish, after the integration of our problem is finished, to go back to our original time x^0:

$$\begin{aligned} dx^0 &= \left(1+\frac{4\mu}{r}\right)dx'^0 \\ &= dx'^0 + \frac{4\mu}{a(1-\varepsilon\cos u)}\sqrt{\frac{a^3}{\mu}}\left(1+\frac{\mu}{2a}\right)(1-\varepsilon\cos u)\,du \\ &= d\left(x'^0 + \frac{4\mu}{a}\sqrt{\frac{a^3}{\mu}}\,u\right). \end{aligned} \tag{5.2.12}$$

After integrating and putting for x'^0 its effective value (5.2.10), we have:

$$x^0 = \sqrt{\frac{a^3}{\mu}}\left[\left(1+\frac{9\mu}{2a}\right)u - \varepsilon\left(1+\frac{\mu}{2a}\right)\sin u\right]. \tag{5.2.13}$$

This result should be substituted instead of (5.2.10c).

In the Newtonian approximation we have:

$$r = a(1-\varepsilon\cos u), \tag{5.2.14a}$$

$$\varphi = 2\arctan\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{u}{2}, \tag{5.2.14b}$$

$$x^0 = \sqrt{\frac{a^3}{\mu}}\,(u-\varepsilon\sin u). \tag{5.2.14c}$$

The question considered in almost every text-book on G. R. T. is that of the path of such a particle. This path and the perihelion

motion can be easily deduced from our more general formulas. We introduce an auxiliary angle ψ:

$$\psi = 2\arctan\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{u}{2}. \tag{5.2.15}$$

It can easily be shown that

$$1-\varepsilon\cos u = \frac{1-\varepsilon^2}{1+\varepsilon\cos\psi}. \tag{5.2.16}$$

Therefore from (5.2.10) we find:

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\left(1-\dfrac{3\mu}{a(1-\varepsilon^2)}\right)\varphi}. \tag{5.2.17}$$

After one rotation, the perihelion has moved by an angle σ, so that

$$\left(1-\frac{3\mu}{a(1-\varepsilon^2)}\right)(2\pi+\sigma) = 2\pi \tag{5.2.18}$$

which gives the famous perihelion motion:

$$\sigma = \frac{6\pi\mu}{a(1-\varepsilon^2)}. \tag{5.2.19}$$

The question now is: what is the physical meaning of all these results? Or, how will they differ in the Galilean coordinate system?

Suppose that the observers at infinity are scattered in a plane parallel to that of the motion. They receive light-signals on a plane for which $Z = x^3$ is very great, but constant. Let us call (x, y) the plane in which the motion proceeds and (X, Y) a far-away plane, but parallel to the (x, y) plane.

If the light-rays fall perpendicular to the (X, Y) plane then putting T for t and denoting

$$\varrho^2 = \xi^s \xi^s \tag{5.2.20}$$

from (5.1.26) we get the connection between the ξ's and the Ξ's:

$$\Xi^1 = \xi^1\left(1+\frac{2\mu}{\varrho}\right), \qquad \Xi^2 = \xi^2\left(1+\frac{2\mu}{\varrho}\right), \tag{5.2.21a}$$

$$\Xi^3 = T-\tau-2\mu\log\frac{2(T-\tau)}{\varrho}. \tag{5.2.21b}$$

Here $\xi^k(\tau)$ $(k = 1, 2)$ is the motion in the (x, y) plane. This motion is observed on the plane $\Xi^3 = \text{const}$. We ask about the connection between T and τ, or rather about the connection between

$$\Delta T = T_2-T_1 \quad \text{and} \quad \Delta\tau = \tau_2-\tau_1 \tag{5.2.22}$$

referring to two events. We wish to find the connection between the objective rhythm of events denoted by T, and the subjective rhythm connected with the choice of the coordinate system. Since $\Xi^3 =$ $=$ const we conclude from (5.2.21b) that

$$\Delta T-\Delta\tau-\frac{2\mu(\Delta T-\Delta\tau)}{T-\tau}+\frac{2\mu}{\varrho}\frac{d\varrho}{d\tau}\Delta\tau = 0. \tag{5.2.23}$$

Since $T-\tau \to \infty$ we have:

$$\Delta T = \Delta\tau\left(1+\frac{2\mu}{\varrho}\frac{d\varrho}{d\tau}\right). \tag{5.2.24}$$

For our solar system $2\mu/\varrho$ is of the order 10^{-8}. The velocity $d\varrho/d\tau$ is very small compared with that of light (remember the choice $c = 1$). Therefore, neglecting expressions of the third order, we can just write:

$$\Delta T = \Delta\tau. \tag{5.2.25}$$

We see, therefore, that for all practical purposes the time has the same rhythm on the objective plane as on the (x, y) plane where the motion takes place.

Now, consider the r, φ coordinates. We have:

$$\Xi^1 = \xi^1\left(1+\frac{2\mu}{\varrho}\right), \qquad \Xi^2 = \xi^2\left(1+\frac{2\mu}{\varrho}\right).$$

Therefore

$$\frac{\Xi^2}{\Xi^1} = \tan\Phi = \frac{\xi^2}{\xi^1} = \tan\varphi. \tag{5.2.26}$$

Of the three elements describing the motion r, φ, x^0 two are the same in the (x, y) as in the (X, Y) plane. What about r? Denoting the projection of r on the (X, Y) plane by R, we have:

$$R = r\left(1+\frac{2\mu}{r}\right) = r+2\mu. \tag{5.2.27}$$

Thus the change in r is a trivial one. It consists in adding to r the constant value 2μ.

Therefore we see the great virtue of the accepted coordinate system in Riemannian space. Only by adding a small but constant value to r and by adding a large but constant value to x^0, can we interpret all our results objectively as those of observers in a Galilean coordinate system who receive the signals of the events through light-rays falling perpendicularly onto a far-removed plane, parallel to the one in which the events take place.

3. THE TWO-BODY PROBLEM

We know that in Newtonian mechanics the solution of the two-body problem is just as easy or as difficult as that of the one-body. In G. R. T. we can also reduce the two-body to the one-body problem, although it is more difficult to do so than in the case of Newtonian dynamics. This statement applies only to the mechanics of finding the solution. It does not apply to the mechanics of finding the Lagrangian; here an essential difference existed between finding it for the one- and for the many-body problem.

Thus, as in the case of the one-body problem, we start from our General Lagrangian which we now apply for two particles. Writing

$$r^2 = (\overset{1}{\xi}{}^s - \overset{2}{\xi}{}^s)(\overset{1}{\xi}{}^s - \overset{2}{\xi}{}^s) \tag{5.3.1}$$

we obtain the following expression:

$$\underset{4}{L}+\underset{6}{L} = \tfrac{1}{2}\overset{1}{\mu}\overset{1}{\xi}{}^{s}{}_{|0}\overset{1}{\xi}{}^{s}{}_{|0}+\tfrac{1}{2}\overset{2}{\mu}\overset{2}{\xi}{}^{s}{}_{|0}\overset{2}{\xi}{}^{s}{}_{|0}+\frac{\overset{1}{\mu}\overset{2}{\mu}}{r}+\tfrac{3}{2}\frac{\overset{1}{\mu}\overset{2}{\mu}}{r}(\overset{1}{\xi}{}^{s}{}_{|0}\overset{1}{\xi}{}^{s}{}_{|0}+\overset{2}{\xi}{}^{s}{}_{|0}\overset{2}{\xi}{}^{s}{}_{|0})+$$

$$-4\frac{\overset{1}{\mu}\overset{2}{\mu}}{r}\overset{1}{\xi}{}^{s}{}_{|0}\overset{2}{\xi}{}^{s}{}_{|0}+\tfrac{1}{8}\overset{1}{\mu}(\overset{1}{\xi}{}^{s}{}_{|0}\overset{1}{\xi}{}^{s}{}_{|0})^{2}+\tfrac{1}{8}\overset{2}{\mu}(\overset{2}{\xi}{}^{s}{}_{|0}\overset{2}{\xi}{}^{s}{}_{|0})+$$

$$-\tfrac{1}{2}\frac{\overset{1}{\mu}\overset{2}{\mu}(\overset{1}{\mu}+\overset{2}{\mu})}{r^{2}}-\tfrac{1}{2}\overset{1}{\mu}\overset{2}{\mu}r_{|\overset{1}{\xi}{}^{s}\overset{2}{\xi}{}^{r}}\overset{1}{\xi}{}^{s}{}_{|0}\overset{2}{\xi}{}^{r}{}_{|0}. \qquad (5.3.2)$$

We shall choose an especially simple coordinate system in which to solve our problem. It will be the system in which the Newtonian center of mass is at rest. This means that we assume:

$$(\underset{2}{\overset{1}{\mu}}+\underset{2}{\overset{2}{\mu}})X^{k} = \underset{2}{\overset{1}{\mu}}\underset{0}{\overset{1}{\xi}}{}^{k}+\underset{2}{\overset{2}{\mu}}\underset{0}{\overset{2}{\xi}}{}^{k} = 0(\lambda^{4}). \qquad (5.3.3)$$

Or: X^k, developed into power series, will start with at least the order two. Therefore the product of $X^k_{|0}X^k_{|0}$ by the mass will be of order at least eight, and can be neglected.

We now introduce a new variable and new constants:

$$\overset{1}{\xi}{}^{s}-\overset{2}{\xi}{}^{s} = \eta^{s}; \quad \mu_0 = \overset{1}{\mu}+\overset{2}{\mu}; \quad \mu = \frac{\overset{1}{\mu}\overset{2}{\mu}}{\mu_0}. \qquad (5.3.4)$$

This, together with (5.3.3), gives:

$$\overset{1}{\xi}{}^{k} = \frac{\mu}{\overset{1}{\mu}}\eta^{k}, \quad \overset{2}{\xi}{}^{k} = -\frac{\mu}{\overset{2}{\mu}}\eta^{k}. \qquad (5.3.5)$$

We can now express L as a function of η and its first derivatives with respect to time. Using

$$r^2 \doteq \eta^s\eta^s = (\overset{1}{\xi}{}^{s}-\overset{2}{\xi}{}^{s})(\overset{1}{\xi}{}^{s}-\overset{2}{\xi}{}^{s}) \qquad (5.3.6)$$

we obtain L/μ, which we again denote by L:

$$\underset{2}{L}+\underset{4}{L} = \tfrac{1}{2}\dot{\eta}^s\dot{\eta}^s+\tfrac{1}{8}\left(1-\frac{3\mu}{\mu_0}\right)(\dot{\eta}^s\dot{\eta}^s)^2+\tfrac{1}{2}(3\mu_0+\mu)\frac{1}{r}\dot{\eta}^s\dot{\eta}^s+$$
$$+\tfrac{1}{2}\frac{\mu}{r^3}(\eta^s\dot{\eta}^s)^2+\frac{\mu_0}{r}-\tfrac{1}{2}\frac{\mu_0^2}{r^2}. \tag{5.3.7}$$

This form of L seems to be much more complicated than that for the one-body case. It can be reduced to the form for one particle, but we must make an appropriate transformation both in time and space. We shall try the following transformation:

$$dx^0 = dx'^0\left(1+\frac{p\mu_0}{r'}\right)\beta \tag{5.3.8a}$$

$$\eta^k = \eta^{k'}\left(1+\frac{s\mu}{r'}\right)\alpha \tag{5.3.8b}$$

where s, p, α, β are constants so chosen that the Lagrangian (5.3.7) takes the form of the Lagrangian for one particle. We here denote:

$$r'^2 = \eta^{s'}\eta^{s'} \tag{5.3.9}$$

and the differentiation with respect to $x^{0'}$ by $0'$. In introducing (5.3.8) into the Lagrangian we must remember that

$$L'dx^{0'} = L\,dx^0 \tag{5.3.10}$$

and that we can multiply the Lagrangian by an arbitrary constant. We finally obtain:

$$\underset{2}{L'}+\underset{4}{L'} = \tfrac{1}{2}\eta^{s'}{}_{|0'}\eta^{s'}{}_{|0'}+\frac{\mu_0}{r'}+\tfrac{1}{2}\left[\tfrac{1}{2}\eta^{s'}{}_{|0'}\eta^{s'}{}_{|0'}-\frac{\mu_0}{r'}\right]^2+\frac{3\mu_0^2}{\left(1-\dfrac{3\mu}{\mu_0}\right)r'^2}. \tag{5.3.11}$$

This form can be achieved only under certain conditions for the

constants a, β, s, p. These are:

$$a = 1-\frac{3\mu}{\mu_0}, \quad \beta = \left(1-\frac{3\mu}{\mu_0}\right)^{3/2},$$
$$s = \tfrac{1}{2}\left(1-\frac{3\mu}{\mu_0}\right)^{-1}, \quad p = \frac{3+2\dfrac{\mu}{\mu_0}}{1-\dfrac{3\mu}{\mu_0}}+1. \tag{5.3.12}$$

Since the expression in brackets in (5.3.11) is constant because the equations of motion in the previous approximation are satisfied, we may write:

$$\underset{2}{L'}+\underset{4}{L'} = \tfrac{1}{2}\dot{\eta}^{s'}\dot{\eta}^{s'}+\frac{\mu_0}{r'}+\frac{3\mu_0^2}{\left(1-3\dfrac{\mu}{\mu_0}\right)r'^2}. \tag{5.3.13}$$

Thus the problem is identical with the one-body problem under the influence of a field with a potential function:

$$V = -\frac{\mu_0}{r'}-\frac{3\mu_0^2}{\left(1-\dfrac{3\mu}{\mu_0}\right)r'^2}. \tag{5.3.14}$$

Therefore the rest of the argument is almost identical with that in the previous section. For the path we find:

$$r' = \frac{a'(1-\varepsilon^2)}{1+\varepsilon\cos\left[\left(1-\dfrac{3\mu_0}{\left(1-\dfrac{3\mu}{\mu_0}\right)a'(1-\varepsilon^2)}\right)\varphi\right]}. \tag{5.3.15}$$

But from (5.3.8b) we find:

$$r = ar'+as\mu = \left(1-\frac{3\mu}{\mu_0}\right)r'+\tfrac{1}{2}\mu. \tag{5.3.16}$$

Therefore, introducing

$$a'\left(1-\frac{3\mu}{\mu_0}\right)=a \tag{5.3.17}$$

we have:

$$r=\frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\left[\left(1-\frac{3\mu_0}{a(1-\varepsilon^2)}\right)\varphi\right]}+\frac{\mu}{2}. \tag{5.3.18}$$

Ignoring the additional constant $\frac{\mu}{2}$, we find that for $\varepsilon<1$ the motion is along an ellipse and that after one rotation the perihelion moves by an angle

$$\sigma=\frac{6\pi\mu_0}{a(1-\varepsilon^2)}, \tag{5.3.19}$$

which is identical with the case of the one-body problem, the only difference being that now μ_0 denotes the sum of the masses.

What is the physical meaning of these results? To answer this question we must go back to the first section of this chapter. At the point $\overset{1}{\xi}{}^s$ and the moment τ a light-ray is emitted. But now the two particles move. Yet the contribution coming from their motion can be neglected, since it is of higher order than two. The other difficulty is that the mass emitting the light-ray is now finite. This, too, will be without influence on the light-ray, because of the spherical symmetry of the emitting particle. Also the fact that the potential is infinite at the point $\overset{1}{\xi}{}^s$ has to be ignored, since the assumption about the existence and character of the singularities is of a purely mathematical character. Indeed, the tweedling process was invented and introduced in order to remove this kind of difficulty from the analytical treatment.

Therefore we may treat the problem of a light-ray emitted at the point $\overset{1}{\xi}{}^s$, at the moment τ, in the same way as we treated the problem in the first and second sections.

Let us slightly generalize the transformation (5.3.4) from $\overset{1}{\xi}$, $\overset{2}{\xi}$, to X^s, η^s, by assuming that the Newtonian center of gravity moves so that at any moment in our coordinate system we have $\overset{2}{\xi}{}^s = 0$. Indeed, such a generalized transformation is:

$$\overset{1}{\xi}{}^s = X^s + \frac{\overset{1}{\mu}}{\mu}\eta^s, \quad \overset{2}{\xi}{}^s = X^s - \frac{\overset{2}{\mu}}{\mu}\eta^s \tag{5.3.20}$$

and the condition

$$\overset{2}{\xi}{}^s = 0 \quad \text{means} \quad X^s = \frac{\overset{2}{\mu}}{\mu}\eta^s. \tag{5.3.21}$$

Therefore

$$\overset{1}{\xi}{}^s = \left(\frac{\overset{1}{\mu}}{\mu} + \frac{\overset{2}{\mu}}{\mu}\right)\eta^s = \eta^s. \tag{5.3.22}$$

This shows the physical meaning of η^s: it is the coordinate of $\overset{1}{\xi}{}^s$ in a system in which the second body is at rest. Therefore the motion of the first body relative to the second or *vice versa* is described by (5.3.18). The light-rays sent by the two bodies are slightly curved and reach a plane parallel to that of the motion. Those from one end, curved by the presence of the particle at the other, are shown in diagram 2. Therefore, if we denote by R the objective distance measured in the infinitely far removed system we have:

$$R = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\left[\left(1-\dfrac{3\mu_0}{a(1-\varepsilon^2)}\right)\varphi\right]} + \frac{\mu}{2} + 2\mu_0. \tag{5.3.23}$$

These are trivial changes in formula (5.3.18) from which the additional $\mu/2$ can be left out as well. Also, as follows from the end of the previous section, the rhythm of time is almost the same

in the Galilean as in our Riemannian coordinate system. Once more we see the virtue of the chosen coordinate system. It gives us almost an objective picture of the motion.

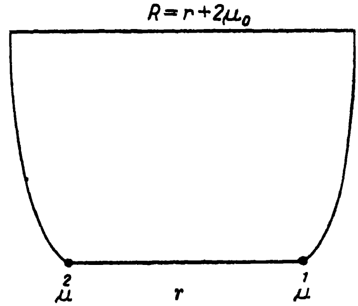

Fig. 2

Distance R judged by an observer in a Galilean coordinate system

4. THE MOTION OF ROTATING BODIES

In the last section of Chapter IV the Lagrangian of a system of two rotating bodies was found. This Lagrangian contained new terms connected with rotation. We already know the corrections to the motion of spherically symmetric non-rotating bodies arising in the post-Newtonian approximation. In the present section, we shall derive the corrections due to rotation, using the Lagrangian

$$L = \tfrac{1}{2}\overset{1}{\mu}\dot{\overset{1}{\xi}}{}^{s}\dot{\overset{1}{\xi}}{}^{s}+\tfrac{1}{2}\overset{2}{\mu}\dot{\overset{2}{\xi}}{}^{s}\dot{\overset{2}{\xi}}{}^{s}+\frac{\overset{1}{\mu}\overset{2}{\mu}}{r}-\frac{2\overset{1}{\mu}}{r^3}\overset{2}{S}{}^{rs}(\dot{\overset{1}{\xi}}{}^{s}-\dot{\overset{2}{\xi}}{}^{s})(\overset{1}{\xi}{}^{r}-\overset{2}{\xi}{}^{r})+$$

$$+2\frac{\overset{2}{\mu}}{r^3}\overset{1}{S}{}^{rs}(\dot{\overset{2}{\xi}}{}^{s}-\dot{\overset{1}{\xi}}{}^{s})(\overset{1}{\xi}{}^{r}-\overset{2}{\xi}{}^{r}), \tag{5.4.1}$$

in which the only post-Newtonian terms are those accounting for rotation and in which the q's from (4.4.21) are neglected.

Using the following notation:

$$
\begin{aligned}
\eta^r &= \overset{1}{\xi}{}^r - \overset{2}{\xi}{}^r, \\
\mu_0 X^r &= \overset{1}{\mu}\overset{1}{\xi}{}^r + \overset{2}{\mu}\overset{2}{\xi}{}^r, \\
\mu_0 &= \overset{1}{\mu} + \overset{2}{\mu}, \qquad\qquad (5.4.2) \\
\mu &= \frac{\overset{1}{\mu}\overset{2}{\mu}}{\overset{1}{\mu} + \overset{2}{\mu}}, \\
\mu S^{rs} &= \overset{1}{\mu}\overset{2}{S}{}^{rs} + \overset{2}{\mu}\overset{1}{S}{}^{rs},
\end{aligned}
$$

we can rewrite the Lagrangian (5.4.1) in the following form:

$$
L = \tfrac{1}{2}\mu_0 \dot{X}^r \dot{X}^r + \mu\left(\tfrac{1}{2}\dot{\eta}^r\dot{\eta}^r + \frac{\mu_0}{r} - \frac{2}{r^3} S^{rs}\dot{\eta}^s\eta^r\right). \qquad (5.4.3)
$$

Since we are considering motion in the system of the Newtonian center of mass, if we omit the constant factor, the following Lagrangian of relative motion is obtained:

$$
L' = \tfrac{1}{2}\dot{\eta}^r\dot{\eta}^r + \frac{\mu_0}{r} - \frac{2}{r^3} S^{rs}\dot{\eta}^s\eta^r. \qquad (5.4.4)
$$

This Lagrangian is invariant with respect to the time displacement transformation (3.4.2) and the rotation

$$
\begin{aligned}
\eta'^r &= \eta^r + M^{rs}\eta^s, \\
S'^{rs} &= S^{rs} + M^{rt}S^{ts} - M^{st}S^{tr} \qquad\qquad (5.4.5)
\end{aligned}
$$

with M^{rs} a constant bivector. Noether's theorem gives:

$$
\tfrac{1}{2}\dot{\eta}^r\dot{\eta}^r - \frac{\mu_0}{r} = E = \text{const} \qquad (5.4.6)
$$

and

$$\dot{J}^{rs} = \frac{2}{r^3}(S^{rt}J^{ts} - S^{st}J^{tr}) \tag{5.4.7}$$

where

$$J^{rs} = \dot{\eta}^r \eta^s - \dot{\eta}^s \eta^r + \frac{2}{r^3}(S^{tr}\eta^t \eta^s - S^{ts}\eta^t \eta^r). \tag{5.4.8}$$

We shall now consider two typical particular cases. First, we assume that the motion is bounded and takes place in the plane perpendicular to the bivector S^{rs}. Written out in vector form, equations (5.4.7) and (5.4.8) are:

$$\dot{\mathbf{J}} = \frac{2}{r^3}\mathbf{S}\times\mathbf{J}, \tag{5.4.9}$$

$$\mathbf{J} = \dot{\boldsymbol{\eta}}\times\boldsymbol{\eta} - \frac{2}{r^3}(\mathbf{S}\times\boldsymbol{\eta})\times\boldsymbol{\eta}, \tag{5.4.10}$$

which, in the case under consideration, gives:

$$J^{rs} = \dot{\eta}^r \eta^s - \dot{\eta}^s \eta^r + \frac{2}{r} S^{rs} = \text{const}, \tag{5.4.11}$$

or

$$\mathbf{J} = \dot{\boldsymbol{\eta}}\times\boldsymbol{\eta} + \frac{2}{r}\mathbf{S} = \text{const}. \tag{5.4.12}$$

Introducing polar coordinates r, φ in the plane of motion we have:

$$J = r^2\dot{\varphi} \pm \frac{2\mu_0 S}{r} \tag{5.4.13}$$

where J and S are the magnitudes of $\mathbf{J}$ and $\mathbf{S}$ respectively and the sign $+$ is for $\mathbf{J}$ and $\mathbf{S}$ pointing in the same direction —

the opposite. The right-hand side of the identity

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = \dot{\eta}^r \dot{\eta}^r (r^2 \dot{\varphi})^{-2}, \qquad u = \frac{1}{r} \tag{5.4.14}$$

can now be expressed as a polynomial in the variable u:

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = J^{-2}\left[2E + \left(2\mu_0 + 8\frac{S}{J}E\right)u + 8\frac{S}{J}\mu_0 u^2\right]. \tag{5.4.15}$$

The solution of the latter equation in the Newtonian approximation is a Kepler ellipse

$$u = p^{-1}[1 + \varepsilon\cos(\varphi - \varphi_0)], \qquad p = \frac{J^2}{\mu_0}, \qquad \varepsilon = \left(1 + \frac{2EJ^2}{\mu_0^2}\right)^{1/2}. \tag{5.4.16}$$

The next approximation yields a rotating ellipse with the very small displacement of the periastron

$$\pm 8\pi \frac{\mu_0 S}{pJ} = \pm 8\pi \frac{\mu_0 S}{J^3} \tag{5.4.17}$$

for each period of the Newtonian motion.

Another case is that of Newtonian motion around a circular orbit $r = a$. Equation (5.4.7) now gives:

$$\dot{\mathbf{J}} = \frac{2}{a^3}\mathbf{S} \times \mathbf{J}. \tag{5.4.18}$$

Thus, the vector $\mathbf{J}$, which is perpendicular to the plane of motion, precesses around the constant vector $\mathbf{S}$ with angular velocity

$$\Omega = \frac{2}{a^3} S \tag{5.4.19}$$

which we can see immediately if we place the $\mathbf{S}$ vector in, say, the x^3 direction and let the $\mathbf{J}$ vector precess around it.

All the considerations concerning the question of measurement are also valid in the case of rotating bodies since the path of a light-

CHAPTER VI

MOTION AND RADIATION

1. A SIMPLE EXAMPLE

In this chapter we intend to show the close connection between the equations of motion and gravitational radiation. To do this we shall start with a simple, almost naive, but instructive example. Let us assume (as we did in the pre-General Relativity era) that we are looking for a field equation of the gravitational field which is consistent with Special Relativity Theory. The simplest idea would be to postulate a scalar D'Alembertian equation, that is one of the form

$$\eta^{\alpha\beta}\varphi_{|\alpha\beta} = -4\pi \sum_{A=1}^{N} \overset{A}{m}\overset{A}{\delta}. \tag{6.1.1}$$

Here, we assume the mass to be a function of time. We also assume

$$\overset{A}{m} = \underset{2}{\overset{A}{m}} + \underset{4}{\overset{A}{m}} + \dots \tag{6.1.2}$$

where $\underset{2}{\overset{A}{m}}$ is constant and throughout the entire chapter we take the velocity of light as one. To find the radiation emitted from the sources of the field which, of course, move arbitrarily, we must first find the retarded solution. We know that such a solution is:

$$\varphi_{\mathrm{RET}}(\mathbf{x}, t) = -\sum_{A}\int d\mathbf{x}' \, \frac{\overset{A}{m}(t-R')\,\delta\left(\mathbf{x}' - \overset{A}{\boldsymbol{\xi}}(t-R')\right)}{R'}, \tag{6.1.3}$$

And for the advanced potential:

$$\varphi_{\mathrm{ADV}}(\mathbf{x},t) = -\sum_{A}\int d\mathbf{x}'\,\frac{\overset{A}{m}(t+R')\,\delta\big(\mathbf{x}'-\overset{A}{\boldsymbol{\xi}}(t+R')\big)}{R'}. \tag{6.1.4}$$

In the last two equations we have:

$$R'^2 = (x'^s-x^s)(x'^s-x^s).$$

Finally we have the standing wave solution:

$$\varphi_{\mathrm{ST}} = \tfrac{1}{2}(\varphi_{\mathrm{RET}}+\varphi_{\mathrm{ADV}}). \tag{6.1.5}$$

Obviously the radiation is connected with the retarded potential solution to which we shall devote our special attention.

We develop the m's and the δ's into power series:

$$m(t-R') = m(t)-\dot{m}(t)R'+\tfrac{1}{2}\ddot{m}(t)R'^2+\dots \tag{6.1.6a}$$

$$\delta\big(\mathbf{x}'-\overset{A}{\boldsymbol{\xi}}(t-R')\big) = \delta\big(\mathbf{x}'-\overset{A}{\boldsymbol{\xi}}(t)\big)-\overset{A}{\delta}_{|0}R'+\tfrac{1}{2}\overset{A}{\delta}_{|00}R'^2-\tfrac{1}{6}\overset{A}{\delta}_{|000}R'^3+\dots \tag{6.1.6b}$$

This gives, up to the fifth order in φ_{RET}:

$$\varphi_{\mathrm{RET}} = -\sum_{A=1}^{N}\big[\underset{2}{\overset{A}{m}}(\overset{A}{R}{}^{-1}+\tfrac{1}{2}\overset{A}{R}_{|00}-\tfrac{1}{6}\overset{A}{R}{}^{2}{}_{|000})-\underset{5}{\dot{m}}\big] \tag{6.1.7}$$

where

$$\overset{A}{R}{}^2 = (x^s-\overset{A}{\xi}{}^s)(x^s-\overset{A}{\xi}{}^s). \tag{6.1.8}$$

This is the usual approach to the field equations (6.1.1) through the retarded potential (6.1.3). But it will be more instructive to approach the problem from the point of view of our approximation method.

To simplify the problem slightly let us assume for the moment that $m = \underset{2}{m}(t)$. In other words, we shall not develop m into a power series. Then, according to our approximation method we have:

$$\underset{2}{\varphi}_{|ss} = 4\pi\sum_{A}\overset{A}{m}\overset{A}{\delta}. \tag{6.1.9}$$

ray depends only on $\underset{2}{h}_{00}$ and $\underset{2}{h}_{sr}$ and rotation does not affect these quantities.

We wish to add some remarks concerning the neglect of the q^{0m} defined by (4.4.15). Of course, they are of the same order as some other expressions in L. Therefore, they should not be neglected. But the situation is this: we have the equations

$$\int \mathfrak{T}^{\alpha\beta}{}_{;\beta}\, d\mathbf{x} = 0 \tag{5.4.20}$$

which for

$$\mathfrak{T}^{\alpha\beta} = \sum_{A=1}^{N} \left(\overset{A}{t}{}^{\alpha\beta}\, \overset{A}{\delta} - \overset{A}{t}{}^{r\alpha\beta}\, \overset{A}{\delta}{}_{|r}\right) \tag{5.4.21}$$

determine $\underset{6}{q}{}^{0m}$ and $\underset{4}{t}{}^{r00}$ in a complicated way. The motion is still arbitrary and determined by our equation

$$\delta \int L\, dt = 0 \tag{5.4.22}$$

in which variation has to be performed with respect to ξ^k. But, to determine L, we must know q^{0m} and this requires the solution of an integral-differential equation. We can, however, find a particular solution of (5.4.20) and (5.4.22). It is:

$$\overset{A}{t}{}^{r00} = \tfrac{1}{2} \overset{A}{S}{}^{rs}\, \dot{\overset{A}{\xi}}{}^{s},$$

$$\int_{t_0}^{t} \Bigl(\overset{A}{\underset{5}{t}}{}^{0r} - \underbrace{\overset{A}{t}{}^{00}\, \dot{\overset{A}{\xi}}}\Bigr)\, dt = -\tfrac{1}{2} \overset{A}{S}{}^{rs}\, \dot{\overset{A}{\xi}}{}^{s}.$$

In this case, the entire reasoning so far presented remains almost unchanged. Only the definition of S^{rs} in the last equation of (5.4.2) changes. This means that for a coordinate system in which the center of gravity is at rest, we have motion of the same character as considered in this section.

Therefore:

$$\underset{2}{\varphi} = -\sum_A \frac{\overset{A}{m}}{\overset{A}{R}}. \tag{6.1.10}$$

From this we find

$$\underset{4}{\varphi}_{|ss} = -\sum_A (\overset{A}{m}\overset{A}{R}^{-1})_{|00} \tag{6.1.11a}$$

and

$$\underset{4}{\varphi} = -\tfrac{1}{2}\sum_A (\overset{A}{m}\,\overset{A}{R})_{|00}. \tag{6.1.11b}$$

Or, if the order of φ jumps by two, we find:

$$\varphi_{\mathrm{ST}} = -\sum_A \sum_{l=0}^{\infty} \frac{1}{(2l)!} \frac{d^{2l}}{dt^{2l}} (\overset{A}{m}\overset{A}{R}^{2l-1}). \tag{6.1.12}$$

But we see immediately that this is the standing wave solution, since it is insensitive to the transformation $t' = -t$. Thus, the simplest solution by our approximation method corresponds to the standing wave solution. To obtain the retarded potential solution we must start with $\underset{3}{\varphi}$ and then again jump by two steps. We must start with a harmonic function and (6.1.7) tells us what kind of harmonic function to take. We must assume

$$\underset{3}{\varphi} = \underset{3}{\dot{m}}. \tag{6.1.13}$$

Then, again proceeding by steps, we find the radiation part of φ:

$$\varphi_{\mathrm{RAD}} = \sum_A \sum_{l=0}^{\infty} \frac{1}{(2l+1)!} \frac{d^{2l+1}}{dt^{2l+1}} (\overset{A}{m}\overset{A}{R}^{2l}). \tag{6.1.14}$$

Thus the approximation method gives us a natural division into the standing wave solution and that representing radiation. From

this we have:

$$\varphi_{\text{RET}} = \varphi_{\text{ST}} + \varphi_{\text{RAD}}, \tag{6.1.15}$$

$$\varphi_{\text{ADV}} = \varphi_{\text{ST}} - \varphi_{\text{RAD}}. \tag{6.1.16}$$

We are returning now to our problem of finding the radiation with the help of the retarded potentials as expressed — up to the fifth order — by (6.1.7).

The energy-momentum tensor for such a field is:

$$4\pi T_\alpha^{\ \beta} = \varphi_{|\alpha}\varphi^{|\beta} - \tfrac{1}{2}\delta_\alpha^\beta \varphi_{|\varrho}\varphi^{|\varrho}. \tag{6.1.17}$$

Indeed, we find:

$$4\pi T_{\alpha\ |\beta}^{\ \beta} = \varphi_{|\beta}^{\ \ \beta}\varphi_{|\alpha} + \varphi^{|\beta}\varphi_{|\alpha\beta} - \varphi^{|\varrho}\varphi_{|\varrho\alpha} = -4\pi \sum_A \overset{A}{m}\overset{A}{\delta}\varphi_{|\alpha}. \tag{6.1.18}$$

Therefore, outside the singularities we always have:

$$T_{\alpha\ |\beta}^{\ \beta} = 0. \tag{6.1.19}$$

As usual, we define the flux of radiation by

$$\int_\Sigma T_\alpha^{\ s} n_s \, dS = \dot{P}_\alpha \tag{6.1.20}$$

where Σ is a very great sphere surrounding all the singularities. We wish to find the energy-momentum flux, up to the eighth order, in the case of our field equations.

We return to our expression (6.1.7). We are interested in it only for very great values of the R's. Let us call r the distance from a fixed chosen point which is the origin of the coordinate system:

$$r^2 = x^s x^s. \tag{6.1.21}$$

Then

$$f(\overset{A}{R}) = f(r) - \frac{df}{dr} r_{|s}\overset{A}{\xi}{}^s + \frac{1}{2}\frac{d^2 f}{dr^2} r_{|sp}\overset{A}{\xi}{}^s\overset{A}{\xi}{}^p - \frac{1}{6}\frac{d^3 f}{dr^3} r_{|spq}\overset{A}{\xi}{}^s\overset{A}{\xi}{}^p\overset{A}{\xi}{}^q + \dots \tag{6.1.22}$$

Since r does not depend on time, we have from (6.1.7) and the last equation:

$$\varphi = -\frac{M}{r}+MX^s\left(\frac{1}{r}\right)_{|s}-\tfrac{1}{2}D^{sp}\left(\frac{1}{r}\right)_{|sp}+\tfrac{1}{2}r_{|s}M\ddot{X}^s+$$

$$-\tfrac{1}{4}\ddot{D}^{sp}r_{|sp}-\tfrac{1}{3}M\dddot{X}^s x^s+\tfrac{1}{6}\dddot{D}^{ss}+\dot{\underset{5}{M}} \qquad (6.1.23)$$

where

$$M = \sum_A \overset{A}{m}, \quad MX^s = \sum_A \overset{A}{m}\overset{A}{\xi}{}^s, \quad D^{sp} = \sum_A \overset{A}{m}\overset{A}{\xi}{}^s\overset{A}{\xi}{}^p. \quad (6.1.24)$$

We find for $\varphi_{|m}$, only up to the order r^{-3}:

$$\varphi_{|m} = -M\left(\frac{1}{r}\right)_{|m}+MX^s\left(\frac{1}{r}\right)_{|sm}+\tfrac{1}{2}M\ddot{X}^s r_{|sm}-\tfrac{1}{4}\ddot{D}^{sp}r_{|spm}-\tfrac{1}{3}M\dddot{X}^m. \qquad (6.1.25)$$

We shall now find

$$4\pi T_0{}^m = \varphi_{|0}\varphi^{|m} = -\varphi_{|0}\varphi_{|m}. \qquad (6.1.26)$$

But we are only interested in the expressions that give contributions to the surface integral such that they are of the order $1/r^2$ and such that their surface integral over the infinite sphere does not vanish. In the lowest non-trivial order we obtain:

$$\dot{P}_0 = \int_\Sigma T_0{}^m n_m dS$$

$$= \frac{2}{9}M^2X^s\frac{d^4X^s}{dt^4}-\frac{1}{9}M^2\dot{X}^s\dddot{X}^s-\frac{1}{6}M\frac{d^4D^{ss}}{dt^4}-M\ddot{\underset{6}{M}}, \qquad (6.1.27a)$$

$$\dot{P}_m = \int T_m{}^s n_s dS = \tfrac{1}{3}M^2\dddot{X}^m, \quad (M = \underset{2}{M}). \qquad (6.1.27b)$$

We see from the last equations how intimately the radiation is bound up with the motion. The radiation would have vanished if we had dealt with the standing wave potential. But even for the retarded potential the radiation can be made whatever we

wish by the proper choice of the motion of the center of gravity and by the proper choice of $\underset{4}{M}$. Let us assume (for example) that

$$\dddot{X}^m = 0 \tag{6.1.28}$$

which is even less than assuming uniform motion for the center of gravity. Then we have:

$$\begin{aligned} \dot{P}_0 &= -\left(\ddot{\underset{6}{M}} + \frac{1}{6}\frac{d^4 D^{ss}}{dt^4}\right)M, \\ \dot{P}_k &= 0. \end{aligned} \tag{6.1.29}$$

The entire vector vanishes if

$$\underset{4}{M} = -\frac{1}{6}\ddot{D}^{ss} + \alpha t + \beta, \qquad (\alpha, \beta = \text{const}). \tag{6.1.30}$$

But

$$\ddot{D}^{ss} = \Big(\sum_{A} \overset{A}{m}\,\overset{A}{\xi}{}^s\,\overset{A}{\xi}{}^s\Big)_{|00} = 4T + 2\sum_{A} \overset{A}{m}\,\overset{A}{\xi}{}^s\,\ddot{\overset{A}{\xi}}{}^s, \tag{6.1.31}$$

if

$$T = \tfrac{1}{2}\sum_{A} \overset{A}{m}\,\dot{\overset{A}{\xi}}{}^s\,\dot{\overset{A}{\xi}}{}^s. \tag{6.1.32}$$

In the case of a Newtonian force, for example, we have:

$$\begin{aligned} \sum_{A} \overset{A}{m}\,\overset{A}{\xi}{}^s\,\ddot{\overset{A}{\xi}}{}^s &= \sum \frac{\overset{A}{m}\overset{B}{m}\overset{A}{\xi}{}^s(\overset{B}{\xi}{}^s - \overset{A}{\xi}{}^s)}{r_{AB}^3} = \frac{1}{2}\sum_{A,B}{}' \frac{\overset{A}{m}\overset{B}{m}\,\overset{A}{\xi}{}^s(\overset{B}{\xi}{}^s - \overset{A}{\xi}{}^s)}{r_{AB}^3} + \\ &\quad -\frac{1}{2}\sum_{A,B}{}' \frac{\overset{A}{m}\overset{B}{m}\,\overset{B}{\xi}{}^s(\overset{B}{\xi}{}^s - \overset{A}{\xi}{}^s)}{r_{AB}^3} = -\frac{1}{2}\sum_{A,B}{}' \frac{\overset{A}{m}\overset{B}{m}}{r_{AB}} = V = \text{potential energy}. \end{aligned} \tag{6.1.33}$$

Therefore

$$\ddot{D}^{ss} = 4T + 2V = 2T + 2E \tag{6.1.34}$$

where E is the (constant) energy.

Therefore, in the case of a Newtonian force, $\dot{P}_0$ vanishes if

$$\underset{4}{M} = -\tfrac{1}{3}T. \tag{6.1.35}$$

We developed $T_m{}^n$ up to the seventh order and $T_0{}^m$ up to the eighth. But the general idea is clear. We can annihilate the radiation by a proper choice of the laws of motion. The situation is more complicated in the case of G. R. T. There the equations of motion are determined by the field equations. Yet the general idea of our investigation remains the same: to establish the connection between the equations of motion and those for radiation. There is little sense in considering radiation without the sources; but motion of these sources and therefore the gravitational radiation, too, are determined by the field equations. They are also determined by the chosen coordinate system, the chosen boundary conditions, and the chosen solutions. We shall show that for the coordinate condition and boundary conditions that seem reasonably simple, the radiation vanishes.

2. THE EQUATIONS OF MOTION IN THE FORM OF A SURFACE INTEGRAL

We shall recall here how the equations of motion in G. R. T. were originally formulated in 1938, or rather eleven years later in 1949. Although the formalism here is much simplified and although we are interested only in the general form of these equations, yet the idea remains the same: to express the equations of motion through surface integrals enclosing the sources.

We recall the Einstein field equations written in the form (2.8.12):

$$\mathfrak{G}^{\mu\nu} + 8\pi\mathcal{T}^{\mu\nu} = K^{\mu\alpha\nu\beta}{}_{|\alpha\beta} + N(\mathfrak{G}^{\mu\nu}) + 8\pi\mathcal{T}^{\mu\nu} = 0. \tag{6.2.1}$$

Here

$$K^{\mu\alpha\nu\beta} = K^{\mu\alpha,\nu\beta} = \tfrac{1}{2}(\eta^{\alpha\nu}\gamma^{\mu\beta} + \eta^{\mu\beta}\gamma^{\alpha\nu} - \eta^{\mu\nu}\gamma^{\alpha\beta} - \eta^{\alpha\beta}\gamma^{\mu\nu}) \tag{6.2.2}$$

is the linear part of $\mathfrak{G}^{\mu\nu}$, being antisymmetric in μ, α and ν, β and insensitive to the interchange of $\mu\alpha$ with $\nu\beta$. The expression $N(\mathfrak{G}^{\mu\nu})$ is the non-linear part of $\mathfrak{G}^{\mu\nu}$ which we shall denote by $\Lambda^{\mu\nu}$ and $\mathcal{T}^{\mu\nu}$ is the energy-momentum tensor density. The relation of the γ's to the g's was explained fully in Chapter II. We rewrite (6.2.1):

$$K^{\mu\alpha,\nu\beta}{}_{|\alpha\beta}+\Lambda^{\mu\nu}+8\pi\mathcal{T}^{\mu\nu} = 0. \tag{6.2.3}$$

We shall now prove a lemma which will be used several times later.

LEMMA. We have a function $\mathcal{F}^{(\dots)ks}$, skew-symmetric in the indices k, s and having arbitrary Greek or Latin indices besides, represented by the dots in the brackets. Then we have:

$$\int_{\Sigma}\mathcal{F}^{(\dots)ks}{}_{|s}\,n_k\,dS = 0 \tag{6.2.4}$$

if Σ is an arbitrary closed two-dimensional surface not passing through the singularities of the field. By n_k we understand:

$$n_k = \cos(x^k, \mathbf{n}) \tag{6.2.5}$$

that is, the components of the "normal unit" vector to the surface Σ. The words "normal" and "unit" are used here only in the conventional sense, to designate the corresponding functions of the coordinates, which are implied by these terms in Euclidean geometry.

The proof is simple enough. First we see that the integral (6.2.4) is certainly independent of the shape of the surface as long as the number of singularities enclosed by the surface does not change. This is so, since

$$\mathcal{F}^{(\dots)ks}{}_{|ks} = 0, \tag{6.2.6}$$

and because of Green's Theorem. Let us now write:

$$\mathcal{F}^{(\dots)ks} = \epsilon^{ksr}\mathcal{A}_r^{(\dots)}, \tag{6.2.7}$$

or explicitly:

$$\mathcal{F}^{(\dots)23} = \mathcal{A}_1^{(\dots)}, \quad \mathcal{F}^{(\dots)31} = \mathcal{A}_2^{(\dots)}, \quad \mathcal{F}^{(\dots)12} = \mathcal{A}_3^{(\dots)}. \tag{6.2.8}$$

Then we can write the integral (6.2.4) in the form:

$$\int_{\Sigma}\mathcal{F}^{(\dots)ks}{}_{|s}\,n_k\,dS = \int_{\Sigma}\epsilon^{ksr}\mathcal{A}_{r|s}\,n_k\,dS = \int_{\Sigma}\mathrm{curl}_n\,\vec{\mathcal{A}}\,dS. \tag{6.2.9}$$

But this integral can be changed by Stokes' Theorem into a line integral over the rim of the surface. But the surface is, as we have assumed, a closed one. Therefore its rim is of zero length. Therefore our lemma is proved.

We go back to the field equations. We shall rewrite (6.2.1) putting $\mu = m$:

$$\mathfrak{G}^{m\nu} = K^{mn,\nu\beta}{}_{|n\beta} + \dot{K}^{m0,\nu\beta}{}_{|\beta} + \Lambda^{m\nu}. \qquad (6.2.10)$$

Let us take any two-dimensional closed surface surrounding the A'th singularity. We call such a surface $\overset{A}{\Sigma}$. Then, because outside the singularity Einstein's tensor vanishes, we have, if we take our lemma into account:

$$\int\limits_{\overset{A}{\Sigma}} \dot{K}^{m0,\nu\beta}{}_{|\beta} n_m \, dS + \int\limits_{\overset{A}{\Sigma}} \Lambda^{m\nu} n_m \, dS = 0, \qquad A = 1, \ldots, N. \qquad (6.2.11)$$

These are $4N$ equations valid (again because of our lemma) for arbitrary surfaces, each of them enclosing only one singularity. Because of this arbitrariness of the surface, they cannot give us any relations between the space coordinates of the field. They can give us only relations between the coordinates of the singularities and their time derivatives. Thus these are by definition the equations of motion for the N particles. From them we can obtain the Newtonian and the post-Newtonian equations of motion. This can be done with the help of the "new" approximation method, although the application of the method is much more elaborate than that presented in this book.

Indeed, to find the post-Newtonian equations of motion in this way we should have to calculate $\underset{2}{\gamma}^{00}$, $\underset{3}{\gamma}^{0m}$, $\underset{4}{\gamma}^{mn}$ and $\underset{5}{\gamma}^{0m}$ and then many surface integrals!

But, as we said before, we regard (6.2.11) by definition as the rigorous equations of motion for N particles.

3. THE EQUATIONS OF MOTION IN THE FORM OF A SPACE INTEGRAL

Let us recall now the form of the $4N$ equations of motion for N singularities known from Chapter I:

$$\int\limits_{\overset{A}{\Omega}} \mathfrak{T}^{\mu\nu}{}_{;\nu} d\mathbf{x} = 0, \qquad A = 1, \ldots, N. \tag{6.3.1}$$

As we remember, this leads to:

$$\frac{d}{dt}\left(\overset{A}{\mu}\dot{\overset{A}{\xi}}{}^{\mu}\right) + \overline{\left\{ {\mu \atop \alpha\beta} \right\}}^{A} \overset{A}{\mu}\dot{\overset{A}{\xi}}{}^{\alpha}\dot{\overset{A}{\xi}}{}^{\beta} = 0, \qquad A = 1, \ldots, N. \tag{6.3.2}$$

The transition from the first to the second of these equations is bound up with the expression for $\mathfrak{T}^{\alpha\beta}$ which is:

$$\mathfrak{T}^{\mu\nu} = \sum_{A=1}^{N} \overset{A}{t}{}^{\mu\nu}\overset{A}{\delta} \tag{6.3.3}$$

and

$$\overset{A}{t}{}^{\mu\nu} = \overset{A}{\mu}\dot{\overset{A}{\xi}}{}^{\mu}\dot{\overset{A}{\xi}}{}^{\nu}. \tag{6.3.4}$$

It can be shown that it is sufficient to assume (6.3.3). Indeed, the specific form of $\overset{A}{t}{}^{\mu\nu}$, as expressed in (6.3.4), follows from the field equations. This means: it is sufficient to assume that the energy-momentum tensor is a linear function of the δ's in order to specify its form.

To show this, let us start from the expression:

$$\int\limits_{\overset{A}{\Omega}} \Theta \mathfrak{T}^{\alpha\beta}{}_{;\beta} d\mathbf{x} = 0 \tag{6.3.5}$$

which follows from the Bianchi identities. Here Θ is an arbitrary function continuous on the world line $\overset{A}{\xi}$. We can write (6.3.5):

$$\int\limits_{\overset{A}{\Omega}} \Theta (\overset{A}{t}{}^{\alpha\beta}\overset{A}{\delta})_{;\beta} d\mathbf{x} = 0. \tag{6.3.6}$$

This leads to:

$$\overset{\frown}{A^{\alpha}}\overset{\frown}{\Theta}+\overset{\frown}{A^{\alpha\beta}}\,\overset{\frown}{\Theta_{|\beta}} = 0. \tag{6.3.7}$$

We shall now calculate $\overset{\frown}{A^{\alpha\beta}}$. Because

$$\overset{A}{\delta}_{|0} = -\overset{A}{\delta}_{|s}\,\dot{\overset{A}{\xi}}{}^{s} \tag{6.3.8}$$

we have:

$$\overset{\frown}{A^{\alpha\beta}}\overset{\frown}{\Theta_{|\beta}} = (-t^{\alpha s}+t^{\alpha 0}\,\dot{\xi}^{s})\overset{\frown}{\Theta}_{|s} = 0. \tag{6.3.9}$$

Since $\dot{\xi}^{0} = 1$, this can be written:

$$\overset{\frown}{A^{\alpha\beta}}\,\overset{\frown}{\Theta_{|\beta}} = (-t^{\alpha\beta}+t^{\alpha 0}\,\dot{\xi}^{\beta})\overset{\frown}{\Theta}_{|\beta} = 0. \tag{6.3.10}$$

Because $\overset{\frown}{\Theta}_{|\beta}$ is arbitrary, we have:

$$\overset{\frown}{A^{\alpha\beta}} = (-t^{\alpha\beta}+t^{\alpha 0}\,\dot{\xi}^{\beta}) = 0. \tag{6.3.11}$$

Putting $\alpha = 0$ we find

$$t^{0\beta} = t^{00}\,\dot{\xi}^{\beta}. \tag{6.3.12}$$

Therefore generally

$$t^{\alpha\beta} = t^{00}\,\dot{\xi}^{\alpha}\,\dot{\xi}^{\beta} = \mu\dot{\xi}^{\alpha}\,\dot{\xi}^{\beta} \tag{6.3.13}$$

which is the proof of our theorem. Obviously $\overset{\frown}{A^{\alpha}} = 0$ gives the equations of motion, that is (6.3.2).

4. THE EQUATIONS OF MOTION IN THE FORM OF SPACE AND SURFACE INTEGRALS

We shall introduce here the third and last definition of the equations of motion.

Let us recall that the first definition was based upon integrals taken over the two-dimensional surfaces surrounding one singularity; that the second was based upon integrals taken over a volume surrounding one of the singularities. In the definition which we shall give here, both the surface and the volume integral will appear.

We start again from the equation

$$\mathfrak{G}^{\mu\nu}+8\pi\mathfrak{T}^{\mu\nu} = 0 \tag{6.4.1}$$

where

$$\mathfrak{G}^{\mu\nu} = K^{\mu\alpha,\nu\beta}{}_{|\alpha\beta}+\Lambda^{\mu\nu}. \tag{6.4.2}$$

Because of the skew-symmetry in the indices $\nu\beta$, we have

$$(\mathfrak{G}^{\mu\nu}+8\pi\mathfrak{T}^{\mu\nu})_{|\nu} = (\Lambda^{\mu\nu}+8\pi\mathfrak{T}^{\mu\nu})_{|\nu} = 0. \tag{6.4.3}$$

Here, obviously, the $\Lambda^{\mu\nu}$ containing only non-linear expressions in the γ's is not a tensor. One would be inclined to regard such an equation as the expression for the conservation law, involving the "pseudo" energy-momentum density $\Lambda^{\mu\nu}$ for the gravitational field. For the moment, however, we shall ignore such an interpretation and we shall use our last equation to define the equations of motion with its help. This we shall do by integrating it over $\overset{A}{\Omega}$. Changing the integral over the space derivatives into a surface integral over $\overset{A}{\Sigma}$, we obtain:

$$\int\limits_{\overset{A}{\Sigma}} \Lambda^{\alpha m} n_m\, dS = -\frac{d}{dt}\int\limits_{\overset{A}{\Omega}} (\Lambda^{0\alpha}+8\pi\mathfrak{T}^{0\alpha})\, d\mathbf{x}. \tag{6.4.4}$$

These are again $4N$ differential equations which we define as the equations of motion for N particles. The energy-momentum tensor does not appear on the left-hand side of these equations since it vanishes at the surface $\overset{A}{\Sigma}$.

5. THE CONSEQUENCES OF THE DIFFERENT FORMS OF THE EQUATIONS OF MOTION

We introduced the equations of motion in three different forms. Firstly: the (Σ) form, according to (6.2.11):

$$-\int\limits_{\overset{A}{\Sigma}} \dot{K}^{m0,\nu\beta}{}_{|\beta} n_m\, dS = \int\limits_{\overset{A}{\Sigma}} \Lambda^{m\nu} n_m\, dS. \tag{6.5.1}$$

Secondly: the (Ω) form, according to (6.3.1) and (6.3.2):

$$\int_{\overset{A}{\Omega}} \mathfrak{T}^{\alpha\beta}{}_{;\beta}\,d\mathbf{x} = 0, \tag{6.5.2a}$$

$$\frac{d}{dt}(\overset{A}{\mu}\overset{A}{\dot{\xi}}{}^{\alpha})+\overset{A}{\mu}\overline{\left\{ {\alpha \atop \mu\nu} \right\}}^{A}\overset{A}{\dot{\xi}}{}^{\mu}\overset{A}{\dot{\xi}}{}^{\nu} = 0. \tag{6.5.2b}$$

Thirdly: the $(\Sigma\Omega)$ form, according to (6.4.4):

$$\int_{\overset{A}{\Sigma}} \Lambda^{\alpha m} n_m\,dS = -\int_{\overset{A}{\Omega}} (\dot{\Lambda}^{0\alpha}+8\pi\dot{\mathcal{T}}^{0\alpha})\,d\mathbf{x}. \tag{6.5.3}$$

We can easily show that these three forms are equivalent to each other, assuming that the field equations

$$\mathfrak{G}^{\mu\nu}+8\pi\mathcal{T}^{\mu\nu} = 0 \tag{6.5.4}$$

are satisfied and assuming also

$$\mathcal{T}^{\mu\nu} = \mathfrak{T}^{\mu\nu} = \sum_{A} \overset{A}{\mu}\overset{A}{\dot{\xi}}{}^{\mu}\overset{A}{\dot{\xi}}{}^{\nu}\overset{A}{\delta}. \tag{6.5.5}$$

Let us first investigate the connection between the (Ω) and the $(\Sigma\Omega)$ form. The (Ω) form was derived from the symbolic equation:

$$\mathfrak{T}^{\alpha\beta}{}_{;\beta} = \mathfrak{T}^{\alpha\beta}{}_{|\beta}+\left\{ {\alpha \atop \mu\nu} \right\}\mathfrak{T}^{\mu\nu} = 0, \tag{6.5.6}$$

whereas the $(\Sigma\Omega)$ form derived from the symbolic equation:

$$8\pi\mathcal{T}^{\alpha\beta}{}_{|\beta}+\Lambda^{\alpha\beta}{}_{|\beta} = 0. \tag{6.5.7}$$

From these two equations follows:

$$\frac{1}{8\pi}\Lambda^{\alpha\beta}{}_{|\beta} = \mathcal{T}^{\mu\nu}\left\{ {\alpha \atop \mu\nu} \right\} \tag{6.5.8}$$

as a consequence of the field equations and the Bianchi identities.

Let us now put into the $(\Sigma\Omega)$ form of the equations of motion, instead of $\mathcal{T}^{0\alpha}$, its explicit expression. Then we have:

$$\frac{d}{dt}(\overset{A}{\mu}\overset{A}{\dot{\xi}}{}^{\alpha}) = -\frac{1}{8\pi}\Big[\int\limits_{\overset{A}{\Sigma}} \Lambda^{\alpha m} n_m dS + \int\limits_{\overset{A}{\Omega}} \dot{\Lambda}^{0\alpha} d\mathbf{x}\Big], \tag{6.5.9}$$

or, because of the (Ω) form:

$$\overset{A}{\mu}\overset{A}{\left\{\begin{matrix}\alpha\\ \mu\nu\end{matrix}\right\}}\overset{A}{\dot{\xi}}{}^{\mu}\overset{A}{\dot{\xi}}{}^{\nu} = +\frac{1}{8\pi}\Big[\int\limits_{\overset{A}{\Sigma}} \Lambda^{\alpha m} n_m dS + \int\limits_{\overset{A}{\Omega}} \dot{\Lambda}^{0\alpha} d\mathbf{x}\Big]. \tag{6.5.10}$$

In these equations we see the connection between the "matter" aspect — on the left-hand side — and the "field" aspect — on the right-hand side — both concepts and their connection being well known since the formulation of classical electrodynamics.

We turn towards the more important problem of finding the consequences of the equivalence of the (Σ) and $(\Sigma\Omega)$ forms. We can write both these forms in one single equation, which will be fundamental to our further reasoning:

$$\int\limits_{\overset{A}{\Sigma}} \Lambda^{\alpha m} n_m dS = -\int\limits_{\overset{A}{\Omega}} (\dot{\Lambda}^{0\alpha} + 8\pi\dot{\mathcal{T}}^{0\alpha}) d\mathbf{x} = -\int\limits_{\overset{A}{\Sigma}} \dot{K}^{m0,\alpha\beta}{}_{|\beta} n_m dS. \tag{6.5.11}$$

From the last equation follows:

$$\int\limits_{\overset{A}{\Omega}} (\Lambda^{0\alpha} + 8\pi\mathcal{T}^{0\alpha}) d\mathbf{x} = \int\limits_{\overset{A}{\Sigma}} K^{m0,\alpha\beta}{}_{|\beta} n_m dS + C^{\alpha} \tag{6.5.12}$$

where the C^{α} are constants of integration. Yet it can be shown that because of the field equations, these constants must vanish. Indeed, the field equations (6.2.1) and (6.2.3) give, for $\mu = 0$:

$$\Lambda^{0\nu} + 8\pi\mathcal{T}^{0\nu} = -K^{0\alpha,\nu\beta}{}_{|\alpha\beta} = -K^{0m,\nu\beta}{}_{|\beta m} = K^{m0,\nu\beta}{}_{|\beta m}. \tag{6.5.13}$$

By integrating the last equation over $\overset{A}{\Omega}$ we obtain:

$$\int\limits_{\overset{A}{\Omega}} (\Lambda^{0\nu} + 8\pi\mathcal{T}^{0\nu}) d\mathbf{x} = \int\limits_{\overset{A}{\Sigma}} K^{m0,\nu\beta}{}_{|\beta} n_m dS. \tag{6.5.14}$$

We remember that $K^{m0,\nu\beta}$ contains only linear expressions in the γ's and (6.2.2) allows us to write them out explicitly. They are:

$$K^{m0,0\beta}{}_{|\beta} = \tfrac{1}{2}(-\gamma^{00}{}_{|m}+\gamma^{ms}{}_{|s}), \qquad (6.5.15)$$

$$K^{m0,k\beta}{}_{|\beta} = \tfrac{1}{2}[\delta^{mk}(\gamma^{00}{}_{|0}+\gamma^{0s}{}_{|s})-\gamma^{0k}{}_{|m}-\gamma^{mk}{}_{|0}]. \qquad (6.5.16)$$

6. THE THREE LINEAR MOMENTA

Because of the form of $\mathcal{T}^{0\alpha}$, we can write (6.5.14) in the form:

$$\frac{1}{8\pi}\int\limits_{\overset{A}{\Omega}} \Lambda^{0\alpha}\,d\mathbf{x}+\overset{A}{\mu}\,\dot{\overset{A}{\xi}}{}^{\alpha} = \frac{1}{8\pi}\int\limits_{\overset{A}{\Sigma}} K^{m0,\alpha\beta}{}_{|\beta}\,n_m\,dS. \qquad (6.6.1)$$

We shall call the "vector"

$$\overset{A}{\mu}\dot{\overset{A}{\xi}}{}^{\alpha} = \overset{A}{P}{}^{\alpha}_{(IN)} \qquad (6.6.2)$$

the inertial linear momentum of the A'th particle. This linear momentum depends on the particle, its velocity, and the field at the points through which this particle moves. This is so because the μ's depend on the tweedled field.

We shall call the "vector"

$$\frac{1}{8\pi}\int\limits_{\overset{A}{\Omega}} \Lambda^{0\alpha}\,d\mathbf{x} = \overset{A}{P}{}^{\alpha}_{(F)} \qquad (6.6.3)$$

the field linear momentum around the A'th particle. It will depend on the region over which we integrate.

We shall call the "vector"

$$\frac{1}{8\pi}\int\limits_{\overset{A}{\Sigma}} K^{m0,\alpha\beta}{}_{|\beta}\,n_m\,dS = \overset{A}{P}{}^{\alpha}{}_{(G)} \qquad (6.6.4)$$

the gravitational linear momentum of and around the A'th particle. We can simply write (6.6.1):

$$\overset{A}{P}{}^{\alpha}_{(G)} = \overset{A}{P}{}^{\alpha}_{(IN)} + \overset{A}{P}{}^{\alpha}_{(F)}. \tag{6.6.5}$$

But $\overset{A}{P}{}^{\alpha}_{(G)}$ is defined only through surface integrals of linear expressions of the first derivatives of the γ's, whereas $\overset{A}{P}{}^{\alpha}_{(F)}$ is defined through a space integral of a non-linear function of the γ's.

We should like to know the physical meaning of these three linear momenta, at least in some special cases. To do so we must quote some results obtained before with the help of the approximation method.

We shall make a special assumption: we are dealing with a two-body problem and we are only interested in the 0-component of the last equation. We denote by $\overset{1}{\mu}_{(IN)}$, $\overset{1}{\mu}_{(F)}$, $\overset{1}{\mu}_{(G)}$, the inertial, field, and gravitational mass of the first particle, corresponding obviously, since $\dot{\xi}^0 = 1$, to $P^0_{(IN)}, P^0_{(F)}$ and $P^0_{(G)}$. Similarly, for the second particle, we shall write "2" above the masses. Now we write for any of these masses:

$$\mu = \underset{2}{\mu} + \underset{4}{\mu} \tag{6.6.6}$$

where $\underset{2}{\mu}$ is the mass in the Newtonian and $\underset{4}{\mu}$ in the post-Newtonian approximation. For $\underset{2}{\mu}$ we have simply that in this approximation the inertial and the gravitational masses are equal, both being constant, whereas the field mass equals zero. Now about the calculation of $\underset{4}{\mu}$. The result is, according to (3.3.29):

$$\underset{4}{\overset{1}{\mu}}{}_{(IN)} = \tfrac{1}{2}\underset{2}{\overset{1}{\mu}}\dot{\overset{1}{\xi}}{}^{s}\dot{\overset{1}{\xi}}{}^{s} + \underset{2}{\overset{1}{\mu}}\underset{2}{\overset{2}{\mu}}r_{12}^{-1}. \tag{6.6.7}$$

Here r_{12} is the distance between the two bodies.

Let us now calculate $\underset{4}{\mu}_{(F)}$. A straightforward calculation, not worth repeating in detail, gives:

$$\underset{4}{\Lambda}^{00} = -\tfrac{3}{4}\varphi_{|s}\varphi_{|s}, \qquad \varphi = -\sum_{A=1}^{N} \frac{2\overset{A}{\mu}}{\overset{A}{r}}. \tag{6.6.8}$$

Therefore, in the case of two particles

$$8\pi \underset{4}{\overset{1}{\mu}}_{(F)} = \tfrac{3}{4}\int\limits_{\overset{1}{\Omega}} \varphi_{|ss}\varphi\, d\mathbf{x} - \tfrac{3}{4}\int\limits_{\overset{1}{\Sigma}} \varphi_{|s}\varphi n_s\, dS$$

$$= -12\pi \frac{\overset{1}{\mu}\overset{2}{\mu}}{r_{12}} - \tfrac{3}{4}\int\limits_{\overset{1}{\Sigma}} \varphi_{|s}\varphi n_s\, dS. \tag{6.6.9}$$

From this equation and (6.6.5) we can find $\mu_{(G)}$:

$$\overset{1}{\mu}_{(G)} = \underset{2}{\overset{1}{\mu}} + \tfrac{1}{2}\underset{2}{\overset{1}{\mu}}\dot{\overset{1}{\xi}}{}^{s}\dot{\overset{1}{\xi}}{}^{s} - \frac{1}{2}\frac{\overset{1}{\mu}\overset{2}{\mu}}{r_{12}} - \frac{3}{32\pi}\int\limits_{\overset{1}{\Sigma}} \varphi_{|s}\varphi n_s\, dS. \tag{6.6.10}$$

We see that the gravitational mass, like the field mass, depends on the finite region over which we integrate. Only in the Newtonian approximation is there no difference between the inertial and gravitational masses.

7. THE EQUATION FOR GRAVITATIONAL RADIATION

In Section 4 we obtained the equations of motion from the differential conservation law

$$\left(\mathcal{T}^{\mu\nu} + \frac{1}{8\pi}\Lambda^{\mu\nu}\right)_{|\nu} = 0. \tag{6.7.1}$$

We shall use the same equation from which we obtained the laws of motion in the $(\Sigma\Omega)$ form as a means to define the gravitational

radiation. The only difference between the point of view represented in Section 4 and here consists in the simple fact that now we do not integrate over the neighbourhood of the A'th singularity but over the entire space, which we shall denote by Ω. The infinite spherical surface surrounding such a space we shall call Σ. Inside the surface Σ, we have particles moving according to the laws of motion deduced from those of the field and formulated in the three last sections. We assume that the particles never reach the surface Σ. Thus, integrating (6.7.1) over the entire space, we find:

$$\int_{\Sigma} \Lambda^{\mu m} n_m \, dS = -\int_{\Omega} (\dot{\Lambda}^{\mu 0} + 8\pi \dot{\underline{T}}^{\mu 0}) \, d\mathbf{x}. \tag{6.7.2}$$

We shall regard by definition the left-hand side of this equation, that is

$$\int_{\Sigma} \Lambda^{\mu m} n_m \, dS \tag{6.7.3}$$

as the flux of gravitational radiation. Thus equation (6.7.2), with the help of which we have defined the gravitational radiation, is closely related to the equations of motion. The only difference between (6.7.2) and the equations of motion consists in the region of integration: Σ and Ω instead of $\overset{A}{\Sigma}$ and $\overset{A}{\Omega}$. Equation (6.7.2) is the $(\Sigma\Omega)$ form of the conservation law, in the same way as previously (6.5.3) was the $(\Sigma\Omega)$ form of the equations of motion.

As before, so now we can write the conservation laws in the (Σ) form:

$$\int_{\Sigma} \Lambda^{\mu m} n_m \, dS = -\int_{\Sigma} \dot{K}^{m0,\mu\beta}{}_{|\beta} n_m \, dS, \tag{6.7.4}$$

or, confronting (6.7.2) and (6.7.4) we see that

$$\int_{\Sigma} \Lambda^{\mu m} n_m \, dS = -\int_{\Omega} (\dot{\Lambda}^{0\mu} + 8\pi \dot{T}^{0\mu}) \, d\mathbf{x} = -\int_{\Sigma} \dot{K}^{m0,\mu\beta}{}_{|\beta} n_m \, dS, \tag{6.7 5}$$

or, as before in Chapter VI, Section 5, we conclude:

$$\int_{\Omega} (\Lambda^{0\mu} + 8\pi\mathcal{T}^{0\mu})\,d\mathbf{x} = \int_{\Sigma} K^{m0,\mu\beta}{}_{|\beta}\, n_m\, dS. \tag{6.7.6}$$

Finally we may also write the conservation equations in the (Ω) form:

$$\int_{\Omega} \mathfrak{T}^{\alpha\beta}{}_{;\beta}\,d\mathbf{x} = \sum_{A=1}^{N} \int_{\overset{A}{\Omega}} \mathfrak{T}^{\alpha\beta}{}_{;\beta}\,d\mathbf{x}$$

$$= \sum_{A=1}^{N} \left[\frac{d}{dt}(\overset{A}{\mu}\,\overset{A}{\dot{\xi}}{}^{\alpha}) + \overline{\begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix}}^{A}\,\overset{A}{\dot{\xi}}{}^{\mu}\,\overset{A}{\dot{\xi}}{}^{\nu}\,\overset{A}{\mu}\right] = 0. \tag{6.7.7}$$

All these equations are a consequence of the equations of motion, which in turn are a consequence of the field equations.

We know that the conservation laws in the (Σ) form, that is (6.7.4), are independent of the shape of the surface, as long as this surface does not pass through singularities. Therefore, we may form the two-dimensional surface Σ enclosing all singularities as consisting of small spheres, each of them surrounding one of the singularities and connected with each other by very thin tubes. But because the flux through these thin tubes equals zero, we can write (6.7.4):

$$\int_{\Sigma} (\Lambda^{\mu m} + K^{m0,\mu\beta}{}_{|\beta})\,n_m\,dS = \sum_{A=1}^{N} \int_{\overset{A}{\Sigma}} (\Lambda^{\mu m} + \dot{K}^{m0,\mu\beta}{}_{|\beta})\,n_m\,dS = 0. \tag{6.7.8}$$

Similarly we may write, using the $(\Sigma\Omega)$ form:

$$\sum_{A=1}^{N} \int_{\overset{A}{\Sigma}} \Lambda^{\mu m}\,n_m\,dS = -\sum_{A=1}^{N} \int_{\overset{A}{\Omega}} (\dot{\Lambda}^{\mu 0} + 8\pi\dot{\mathcal{T}}^{\mu 0})\,d\mathbf{x}. \tag{6.7.9}$$

Yet, generally

$$\sum_{A=1}^{N} \int_{\overset{A}{\Sigma}} \Lambda^{\mu m}\,n_m\,dS \neq \int_{\Sigma} \Lambda^{\mu m}\,n_m\,dS. \tag{6.7.10}$$

We regard the right-hand side and not the left-hand side of this inequality as the definition of gravitational radiation. This definition is not precisely the usual one, because $\Lambda^{\mu m}$ also contains derivatives of the second order. But it seems to me more consistent to use for the gravitational radiation this definition which is closely connected with the equations of motion.

Similarly as before we define the total linear momentum:

$$P^{\alpha}_{(IN)} = \int_{\Omega} \mathfrak{T}^{0\alpha} d\mathbf{x} = \sum_{A=1}^{N} \overset{A}{\mu} \overset{\overset{A}{\cdot}}{\xi}{}^{\alpha}, \tag{6.7.11}$$

the total linear field momentum:

$$P^{\alpha}_{(F)} = \frac{1}{8\pi} \int_{\Omega} \Lambda^{0\alpha} d\mathbf{x} \tag{6.7.12}$$

and the total linear gravitational momentum:

$$P^{\alpha}_{(G)} = \frac{1}{8\pi} \int_{\Sigma} K^{m0,\alpha\beta}{}_{|\beta} n_m dS. \tag{6.7.13}$$

Similarly as before, we can write (6.7.6) in the form:

$$P^{\alpha}_{(G)} = P^{\alpha}_{(IN)} + P^{\alpha}_{(F)}. \tag{6.7.14}$$

Here we face a situation that appears at almost every step in electrodynamics. Quantities referring to the world-line as $P^{\alpha}_{(IN}$ does, are connected with those referring to the field, like $P^{\alpha}_{(F)}$ and $P^{\alpha}{}_{(G)}$.

It is instructive to compare the situation as presented here for the gravitational field with that of an electromagnetic field in Special Relativity Theory, in which case we have the electromagnetic energy-momentum tensor $E^{\alpha\beta}$ and the conservation law $E^{\alpha\beta}{}_{|\beta} = 0$. From it by integration over the entire space we find:

$$\int_{\Sigma} E^{\alpha m} n_m dS = -\int_{\Omega} \dot{E}^{\alpha 0} d\mathbf{x}. \tag{6.7.15}$$

The right-hand side is usually understood as the (negative) time derivative of the entire field linear momentum. For the electromagnetic field it is the only linear momentum that appears in the conservation law. In the case in which the left-hand side representing the flux of electromagnetic radiation vanishes, we conclude that the field linear momentum is a constant vector.

The situation is different in the case of a gravitational field. The flux of gravitational radiation is equal to the (negative) time derivative of the sum of two linear momenta: that of the gravitational field and that of inertial matter. In the case in which the gravitational radiation flux vanishes, it is the sum of the two momenta that remains constant. This sum we call the gravitational linear momentum and it is equal to a surface integral of an expression consisting of linear derivatives of the γ's.

To give an example showing the difference between these three total linear momenta let us return once more to the example given at the end of the previous section; that of two particles and their total masses calculated in the Newtonian and post-Newtonian order. As before we have in the Newtonian approximation:

$$\underset{2}{P^0_{(IN)}} = \underset{2}{P^0_{(G)}} = \text{const}; \qquad \underset{2}{P^0_{(F)}} = 0. \tag{6.7.16}$$

Then, in the post-Newtonian approximation, since the surface integral (6.7.10) over the infinitely great surface vanishes,

$$\underset{4}{P^0_{(IN)}} = \text{Total Kinetic Energy} - 2 \text{ Potential Energy}.$$

$$\underset{4}{P^0_{(G)}} = \text{Total Kinetic Energy} + \text{Potential Energy} = \text{const}.$$

$$\underset{4}{P^0_{(F)}} = 3 \text{ Potential Energy}.$$

We see from the expression for $\underset{4}{P^0_{(G)}}$ that in this approximation the surface integral over "the normal component of the Poynting vector" vanishes in this case for gravitational radiation. This is so,

because

$$\underset{2}{\dot{P}}{}^{0}_{(G)} + \underset{4}{\dot{P}}{}^{0}_{(G)} = -\frac{1}{8\pi}\int \Lambda^{0m} n_m\, dS = 0, \tag{6.7.17}$$

that is, because of the Newtonian equations of motion the sum of kinetic and potential energy is constant.

8. ON THE INVARIANCE PROPERTIES OF $P^{\alpha}{}_{(G)}$

We ask: what are the transformation properties of $P^{\alpha}_{(G)}$ under small transformations of the type

$$x^{*\mu} = x^{\mu} + a^{\mu}(x)? \tag{6.8.1}$$

We shall consistently neglect all products of the a's as well the products of the γ's and a's. We recall that, according to (6.7.13), (6.5.15) and (6.5.16):

$$P^{0}_{(G)} = \frac{1}{8\pi}\int_{\Sigma} K^{m0,0\beta}{}_{|\beta} n_m\, dS = \frac{1}{16\pi}\int_{\Sigma} (-\gamma^{00}{}_{|m} + \gamma^{ms}{}_{|s}) n_m\, dS, \tag{6.8.2a}$$

$$P^{k}_{(G)} = \frac{1}{8\pi}\int_{\Sigma} K^{m0,k\beta}{}_{|\beta} n_m\, dS = \frac{1}{16\pi}\int_{\Sigma} (\delta^{mk}\gamma^{0\alpha}{}_{|\alpha} - \gamma^{k0}{}_{|m} - \gamma^{mk}{}_{|0}) n_m\, dS. \tag{6.8.2b}$$

We start by investigating the changes caused in $P^{0}_{(G)}$ by transformation (6.8.1). Thus we have to use the formula:

$$\mathfrak{g}^{*\mu\nu} = \mathfrak{g}^{\alpha\beta} x^{*\mu}{}_{|\alpha} x^{*\nu}{}_{|\beta} \det\frac{\partial x}{\partial x^{*}}. \tag{6.8.3}$$

Into this equation we have to introduce:

$$x^{*\mu}{}_{|\alpha} = \delta^{\mu}{}_{\alpha} + a^{\mu}{}_{|\alpha}, \tag{6.8.4a}$$

$$\det\frac{\partial x}{\partial x^{*}} = 1 - a^{\alpha}{}_{|\alpha}. \tag{6.8.4b}$$

From the last three equations it follows (since $g^{\mu\nu} = \eta^{\mu\nu}+\gamma^{\mu\nu}$) that

$$\gamma^{*00} = \gamma^{00}+a^{0}{}_{|0}-a^{s}{}_{|s}, \tag{6.8.5a}$$

$$\gamma^{*0m} = \gamma^{0m} - a^{0}{}_{|m}+a^{m}{}_{|0}, \tag{6.8.5b}$$

$$\gamma^{*mn} = \gamma^{mn}-a^{m}{}_{|n}-a^{n}{}_{|m}+\delta^{mn}a^{s}{}_{|s}+\delta^{mn}a^{0}{}_{|0}. \tag{6.8.5c}$$

Introducing these into (6.5.15) and (6.5.16) we find:

$$K^{*m0,0\beta}{}_{|\beta} = K^{m0,0\beta}{}_{|\beta}+(a^{s}{}_{|m}-a^{m}{}_{|s})_{|s}, \tag{6.8.6a}$$

$$K^{*m0,k\beta}{}_{|\beta} = K^{m0,k\beta}{}_{|\beta}+(\delta^{ks}a^{0}{}_{|m}-\delta^{mk}a^{0}{}_{|s})_{|s}+(\delta^{ks}a^{m}{}_{|0}-\delta^{mk}a^{s}{}_{|0})_{|s}. \tag{6.8.6b}$$

The bracketed expressions on the right-hand side of (6.8.6) are skew-symmetric in the indices m, s; therefore, because of the lemma proved in Section 2, we have:

$$\int_{\Sigma} K^{*m0,\nu\beta}n_m\,dS = \int_{\Sigma} K^{m0,\nu\beta}n_m\,dS, \tag{6.8.7}$$

or

$$P^{*\alpha}_{(G)} = P^{\alpha}_{(G)},$$

which shows that the components of the gravitational linear momentum are invariant under small transformations of the coordinate system. This would not be so if the products of the γ's and a's were taken into account. We can see this immediately in the special case in which the gravitational flux vanishes. There, with respect to a Lorentz transformation the $\Lambda^{0\alpha}$ and $T^{0\alpha}$ behave like tensor components, and therefore $P^{\alpha}_{(G)}$ like a constant vector. Thus with respect to a Lorentz transformation the $P^{\alpha}_{(G)}$ are not invariants.

9. GRAVITATIONAL RADIATION AND THE CHOICE OF A COORDINATE SYSTEM

We arrive now at the principal question: can we find a reasonable coordinate system for which the gravitational radiation vanishes? In other words: is the gravitational radiation as defined here something that can be annihilated by a proper choice of a coordinate

system, or is it something that has an absolute meaning? This means: can we find a coordinate system for which $\dot{P}^{\alpha}_{(G)} = 0$ and $P^{\alpha}{}_{(G)} = \text{constant}$, that is for which the law of conservation of the sum of the two linear momenta — the inertial and the field — is valid?

Before we answer this question let us recall once more the definition of $\dot{P}^{\alpha}_{(G)}$. It was given in (6.7.13):

$$\dot{P}^{\alpha}_{(G)} = \frac{1}{8\pi} \int_{\Sigma} \dot{K}^{m0,\alpha\beta}{}_{|\beta} n_m dS. \tag{6.9.1}$$

But because of (6.7.8) we can also write:

$$\dot{P}^{\alpha}_{(G)} = -\frac{1}{8\pi} \int_{\Sigma} \Lambda^{\alpha m} n_m dS. \tag{6.9.2}$$

We can, therefore, calculate the change in linear momenta in two ways: first, through the linear expressions in the γ's, that is through (6.9.1), second through the non-linear expressions in the γ's, that is through (6.9.2).

Now about the choice of our coordinate system: all our assumptions will concern only the behaviour of our coordinate system and the γ's for $r \to \infty$, r being the "distance" from a fixed point, that is $r^2 = x^k x^k$ for very large r's. We assume that for $r \to \infty$ our metric becomes Galilean at least like $\log r / r^{\sigma}$, with $\sigma > 0$, that is:

$$|\gamma^{\alpha\beta}| \leqslant \left| A^{\alpha\beta}(t) \cdot \frac{\log r}{r^{\sigma}} \right|. \tag{6.9.3}$$

Furthermore we assume for $r \to \infty$ the following coordinate conditions:

$$\gamma^{0\alpha}{}_{|\alpha} = O\left(\frac{1}{r^{2+\sigma}}\right), \quad (\sigma > 0), \tag{6.9.4a}$$

$$\gamma^{mn}{}_{|n} = O\left(\frac{1}{r^{2+\sigma}}\right). \tag{6.9.4b}$$

We should like to draw attention to the condition (6.9.4b) which is different from de Donder's condition $\gamma^{ma}{}_{|a} = 0$ used so extensively by Fock. The results presented here are essentially due to the choice of the coordinate condition (6.9.4b) which takes for granted the ideas on which the "new approximation method" is being built: that space and time have to be treated differently; this difference manifests itself in the choice of our coordinate conditions. Indeed, it was this coordinate condition that was chosen in earlier papers by Einstein, Infeld and Hoffmann. The chief difference between this work and that of other authors, notably Trautman, also consists in the different coordinate condition used here.

Now for this special coordinate condition, the $K^{m0,\nu\beta}{}_{|\beta}$ take a very simple form:

$$K^{m0,0\beta}{}_{|\beta} = -\tfrac{1}{2}\gamma^{00}{}_{|m}, \tag{6.9.5a}$$

$$K^{m0,k\beta}{}_{|\beta} = -\tfrac{1}{2}(\gamma^{k0}{}_{|m}+\gamma^{mk}{}_{|0}), \tag{6.9.5b}$$

as follows from (6.8.2). Also the field equations take on a simpler form in this coordinate system. At this moment we are only interested in their $(0, 0)$ and $(0, k)$ components, and only for $r \to \infty$, where we can forget the energy-momentum tensor:

$$K^{0a,0b}{}_{|ab}+\Lambda^{00} = 0, \tag{6.9.6a}$$

$$K^{0a,k\beta}{}_{|a\beta}+\Lambda^{0k} = 0, \tag{6.9.6b}$$

or because of (6.9.5) we have in our coordinate system for $r \to \infty$) simply:

$$\tfrac{1}{2}\gamma^{00}{}_{|ss} = -\Lambda^{00}, \tag{6.9.7a}$$

$$\tfrac{1}{2}\gamma^{0k}{}_{|ss} = -\Lambda^{0k}. \qquad (6.9.7b,$$

Let us concentrate for a while on the second of these equations. As we know, Λ^{0k} is a product of the γ's and their derivatives. If the γ's are of the order $1/r$, the derivatives are of the order $1/r^2$ or higher in $1/r$. Therefore Λ^{0k} must be at least of second order in $1/r$ It would seem, therefore, that Λ^{0m} can make a contributionto. the surface integral (6.9.2), and therefore that $P^0_{(G)}$ can be a function of time in this coordinate system. However this is not so.

We develop the essential expressions in Λ^{0m}, which may give a contribution to the surface integral (6.9.2), into a power series in x^s/r, assuming, of course, that this is possible. That is:

$$\Lambda^{0m} = \frac{a^{0m}}{r^2} + \frac{a_s^{0m} x^s}{r^3} + \frac{a_{sp}^{0m} x^s x^p}{r^4} + \ldots \tag{6.9.8}$$

where the coefficients a^{0m} etc. are functions of time only. Since, however, the left-hand side of (6.9.7b) is a pure Laplacian it follows that the corresponding γ^{0m} must be of the form:

$$\gamma^{0m} = A^{0m} \log r + A^{0m}{}_s \frac{x^s}{r} + \ldots \tag{6.9.9}$$

Here the A^{0m} etc. are such functions of the time that the equations (6.9.7b) are satisfied. But that is in contradiction to our assumption that the γ^{0m}, like all the other γ's, become Galilean at infinity. Therefore Λ^{0m} can have no expressions of the order r^{-2}; it must start with a higher order in $1/r$; therefore $\dot{P}^0_{(G)}$ must equal zero, therefore $P^0_{(G)}$ must be constant!

This result is bound very definitely to the choice of our coordinate system. It forces the appearance of a Laplacian, instead of a d'Alembertian on the left-hand side of (6.9.7b). Indeed in the case of a d'Alembertian we could have had expressions for γ^{0m} o. the order a/r and a/r^2 which are consistent with Λ^{0m} of the order r^{-2}f

Turning to the definition of $P^0_{(G)}$ as the integral of the linear terms of the γ's we see that

$$\dot{P}^0_{(G)} = \frac{1}{8\pi} \int_{\Sigma} \dot{K}^{m0,0\beta}{}_{|\beta} n_m \, dS = -\frac{1}{16\pi} \int_{\Sigma} \gamma^{00}{}_{|m0} n_m \, dS = 0. \tag{6.9.10}$$

Let us assume that

$$\gamma^{00} = \frac{4M}{r} + \text{terms of the higher order in } \frac{1}{r}. \tag{6.9.11}$$

Then, by taking the surface integral, we find:

$$P^0_{(G)} = -\frac{1}{16\pi} \int \left(\frac{4M}{r}\right)_{|m} n_m \, dS = M. \tag{6.9.12}$$

Therefore M must be constant and is equal to the total gravitational mass.

We shall now turn towards the $P^k_{(G)}$. We shall prove that we may further specialize our coordinate system at infinity so that they, too, will be constant. That means: the coordinate transformation that we shall introduce now will not change the essential features of the previous coordinate system. By this we mean that the γ^*'s will vanish at infinity like $\log r/r^a$ $(a > 0)$ and that ${\gamma^{*0\alpha}}_{|\alpha}$ and ${\gamma^{*mn}}_{|n}$ will go to zero at least like $1/r^{2+a}$ with $a > 0$.

For the sake of simplicity we shall make a further assumption: that for $r \to \infty$, $\gamma^{0\alpha}$ goes to zero like $1/r$. We could have stuck to our assumption, that $\gamma^{0\alpha}$ behaves like $\log r/r^a$ with $a > 0$ for $r \to \infty$, but our argument would then be slightly more complicated.

We introduce the coordinate transformation:

$$x^{*k} = x^k + \frac{C^k}{r} \tag{6.9.13}$$

where the C^k are functions of t only. We shall, however, in transforming the γ's write out all expressions, linear, or non-linear, up to the order of $1/r^{2+a}$ with $a > 0$. From (6.8.3) and (6.9.13) we find:

$$\gamma^{*00} = \gamma^{00} - C^s\left(\frac{1}{r}\right)_{|s} + O\left(\frac{1}{r^{2+a}}\right), \tag{6.9.14a}$$

$$\gamma^{*0m} = \gamma^{0m} + \dot{C}^m\frac{1}{r} + \dot{C}^m\frac{\gamma^{00}}{r} + O\left(\frac{1}{r^{2+a}}\right), \tag{6.9.14b}$$

$$\begin{aligned}\gamma^{*mn} = \gamma^{mn} - C^m\left(\frac{1}{r}\right)_{|n} - C^n\left(\frac{1}{r}\right)_{|m} + \delta^{mn}C^s\left(\frac{1}{r}\right)_{|s} + \\ + \dot{C}^m\frac{\gamma^{0n}}{r} + \dot{C}^n\frac{\gamma^{0m}}{r} + \dot{C}^m\dot{C}^n\frac{1}{r^2} + O\left(\frac{1}{r^{2+a}}\right).\end{aligned} \tag{6.9.14c}$$

Since in (6.9.14b) ${\gamma^{00}}_{|m}/r$ is, at least, of the order $1/r^{2+a}$ (with $a > 0$) we see that for $r \to \infty$:

$${\gamma^{*00}}_{|0} + {\gamma^{*0m}}_{|m} = O\left(\frac{1}{r^{2+a}}\right), \quad a > 0, \tag{6.9.15}$$

and similarly

$$\gamma^{*mn}{}_{|n} = O\left(\frac{1}{r^{2+\alpha}}\right), \tag{6.9.16}$$

since γ^{0m} is of the order $1/r$ and its space derivatives of the order $1/r^2$. We know that the linear terms in (6.9.14) do not have any influence upon $P^k_{(G)}$. This was shown explicitly in the previous section. Furthermore, no terms of the order $1/r^{2+\alpha}$ have any influence upon the surface integral. Finally since in (6.9.5b) γ^{*0k} is differentiated with respect to space, it is sufficient to take into account in γ^{*0k} only the expressions of the order $1/r$. We see, too, that in (6.9.14c) the expression $\dot{C}^m\dot{C}^n/r^2$ cannot make any contribution to the surface integral, because it is multiplied by $n_m = x^m/r$ and the surface integral vanishes because of the non-symmetry. Thus for the sake of finding $P^{*k}_{(G)}$, we may assume:

$$\gamma^{*00} = \gamma^{00}, \tag{6.9.17a}$$

$$\gamma^{*0m} = \gamma^{0m}, \tag{6.9.17b}$$

$$\gamma^{*mn} = \gamma^{mn} + \dot{C}^m\frac{\gamma^{0n}}{r} + \dot{C}^n\frac{\gamma^{0m}}{r}. \tag{6.9.17c}$$

We shall introduce now the following functions of time:

$$A^n_m = \frac{1}{16\pi}\int\limits_{\Sigma}\frac{\gamma^{0n}}{r}n_m dS. \tag{6.9.18}$$

Therefore, because of (6.9.1) and (6.9.5b) we have:

$$\dot{P}^{*k}_{(G)} = \dot{P}^k_{(G)} - (A^k_m\dot{C}^m + A^s_s\dot{C}^k)_{|00}. \tag{6.9.19}$$

Thus, generally, we can, without invalidating our original conditions, annihilate $\dot{P}^{*k}_{(G)}$. To do this we must solve the differential equation:

$$\dot{P}^k_{(G)} = (A^k_m\dot{C}^m + A^s_s\dot{C}^k)_{|00}. \tag{6.9.20}$$

In it $P^k_{(G)}$ and A^k_m are known functions of time. From them we find the C's and introduce with their help the coordinate transformation (6.9.13). In the new "starred" coordinate system we have:

$$\dot{P}^{*a}_{(G)} = 0, \qquad P^{*a}_{(G)} = \text{const.} \tag{6.9.21}$$

Thus in such a coordinate system the sum of the inertial and field linear momenta remains constant!

(In the case in which for $r \to \infty$ we have $\gamma^{0m} = A^{0m}(t)\,\dfrac{\log r}{r^a}$, we would have to replace (6.9.13) by

$$x^{*k} = x^k + C^k \frac{r^{a-2}}{\log r}.\Big)$$

As far as we can see, there is only one argument that could be levelled against this reasoning: what happens if A^k_m vanishes?

This question can easily be answered by a generalization of the argument just presented.

Indeed let us introduce instead of the transformation

$$x^{*k} = x^k + \frac{C^k}{r} \tag{6.9.22}$$

a slightly more general transformation

$$x^{*k} = x^k + a^k \tag{6.9.23}$$

where a^k has the form:

$$a^k = \frac{b^k}{r} + \frac{b^k_s x^s}{r^2} + \ldots \tag{6.9.24}$$

and the b's in (6.9.24) are functions of t only. In other words we assume: a^k is of the order r^{-1} and does not change its order by a time derivative but lowers it by a space derivative. Thus we have:

$$\gamma^{*00} = \gamma^{00} - a^s{}_{|s} + O\left(\frac{1}{r^{2+a}}\right), \tag{6.9.25a}$$

$$\gamma^{*0m} = \gamma^{0m} + a^m{}_{|0} + a^m{}_{|0}\gamma^{00} + O\left(\frac{1}{r^{2+\alpha}}\right), \qquad (6.9.25b)$$

$$\gamma^{*mn} = \gamma^{mn} - a^m{}_{|n} - a^n{}_{|m} + \delta^{mn}a^s{}_{|s} + a^m{}_{|0}\gamma^{0n} + $$
$$+ a^n{}_{|0}\gamma^{0m} + a^n{}_{|0}a^m{}_{|0} + O\left(\frac{1}{r^{2+\alpha}}\right). \qquad (6.9.25c)$$

Since only products of the type $\gamma\cdot a$ and $a\cdot a$ can make contributions to $P^k_{(G)}$ and then only if they are of the proper order, we may assume (in order to find $P^k_{(G)}$), similarly as in (6.9.17):

$$\gamma^{*00} = \gamma^{00}, \qquad (6.9.26a)$$

$$\gamma^{*0m} = \gamma^{0m}, \qquad (6.9.26b)$$

$$\gamma^{*mn} = \gamma^{mn} + a^m{}_{|0}\gamma^{0n} + a^n{}_{|0}\gamma^{0m} + a^m{}_{|0}a^n{}_{|0}. \qquad (6.9.26c)$$

Therefore, because of (6.8.2b):

$$P^{*k}_{(G)} = P^k_{(G)} - \frac{1}{16\pi}\int_{\Sigma}(a^k{}_{|0}\gamma^{0m} + a^m{}_{|0}\gamma^{0k} + a^k{}_{|0}a^m{}_{|0})_{|0}n_m dS. \qquad (6.9.27)$$

Thus we may annihilate $P^{*k}_{(G)}$ by any transformation satisfying

$$P^k_{(G)} = \frac{1}{16\pi}\int_{\Sigma}(a^k{}_{|0}\gamma^{0m} + a^m{}_{|0}\gamma^{0k} + a^k{}_{|0}a^m{}_{|0})_{|0}n_m dS. \qquad (6.9.28)$$

Even if γ^{0m} is of the order r^{-2}, such a's will always exist. In such a special case we have:

$$P^k_{(G)} = \frac{1}{16\pi}\int_{\Sigma}(a^m{}_{|0}a^k{}_{|0})_{|0}n_m dS. \qquad (6.9.29)$$

It is sufficient to choose the a's of the form:

$$a^k = \frac{\alpha^k}{r} + \frac{\beta x^k}{r^2}, \qquad (6.9.30)$$

then we get the equation for α^k and β:

$$P^k_{(G)} = \tfrac{1}{3}(\dot{\alpha}^k\dot{\beta})_{|0}.$$

This transformation leaves the coordinate conditions essentially invariant as we see from (6.9.25); that is we have:

$$\gamma^{*0\alpha}{}_{|\alpha} = O\left(\frac{1}{r^{2+\alpha}}\right), \tag{6.9.31a}$$

$$\gamma^{*mn}{}_{|n} = O\left(\frac{1}{r^{2+\alpha}}\right). \tag{6.9.31b}$$

10. ON THE GENERALIZATION OF THE COORDINATE SYSTEM

Until now we have assumed that for the metric field at infinity:

1. $$|\gamma^{\alpha\beta}| \leqslant \left| M^{\alpha\beta}(t)\frac{\log r}{r^\sigma}\right|, \quad \sigma > 0; \tag{6.10.1}$$

2. $$\gamma^{0\alpha}{}_{|\alpha} = O\left(\frac{1}{r^{2+\sigma}}\right), \quad \gamma^{mn}{}_{|n} = O\left(\frac{1}{r^{2+\sigma}}\right); \tag{6.10.2}$$

3. Λ^{0m}, as far as it gives a contribution to the surface integral, can be developed for $r \to \infty$ into a power series in x^s/r, the coefficients being functions of time.

We proved that under these conditions:

A. $P^0_{(G)}$ is constant; as a matter of fact to prove this we did not need to assume 1. It was sufficient to assume that

$$|\gamma^{0m}| \leqslant \left| M^{0m}(t)\,\frac{\log r}{r^a}\right| \quad \text{with} \quad a > 0.$$

B. The $P^a_{(G)}$ are invariant under a transformation

$$x^{*a} = x^a + a^a \tag{6.10.3}$$

if for $r \to \infty$ we neglect all products of $\gamma \cdot a$ and $a \cdot a$.

We shall call

$$x^{*a} = x^a + a^a \tag{6.10.3}$$

a proper transformation if, for $r \to \infty$ the following conditions are satisfied: a^a is of the order $1/r$; the space derivative lowers the

order by one, whereas the time derivative leaves the order unchanged; or: a^α can be developed into a power series in x^s/r with coefficients depending on time only. Furthermore we assume that for a proper transformation the $K^{*m0,\alpha\beta}{}_{|\beta}$ are to be calculated up to the order $1/r^{2+\sigma}$ with $\sigma > 0$; that is if necessary the non-linear terms in the transformation of $\gamma^{\alpha\beta}$ should be taken into account.

With such a definition of the proper transformation we have proved:

C. $P^0_{(G)}$ and the coordinate conditions (6.10.2) are invariant under a proper space transformation, that is if:

$$x^{*0} = x^0, \tag{6.10.4a}$$

$$x^{*k} = x^k + a^k. \tag{6.10.4b}$$

Also we proved:

D. We can always choose such a's in the proper transformation (6.10.4) that

$$P^k_{(G)} = 0. \tag{6.10.5}$$

Now we ask: what happens to $P^0_{(G)}$ and the coordinate conditions (6.10.2) if we use not the proper transformation (6.10.4), but the full proper transformation (6.10.3)? To answer this question it is sufficient to consider only the proper time transformation, that is:

$$x^{*0} = x^0 + a^0. \tag{6.10.6}$$

We have:

$$\gamma^{*00} = \gamma^{00} + a^0{}_{|0} + a^0{}_{|0}\gamma^{00} + O\left(\frac{1}{r^{2+a}}\right), \qquad a > 0, \tag{6.10.7}$$

$$\gamma^{*0m} = \gamma^{0m} - a^0{}_{|m} + O\left(\frac{1}{r^{2+a}}\right), \tag{6.10.8}$$

$$\gamma^{*mn} = \gamma^{mn} + \delta^{mn} a^0{}_{|0} + O\left(\frac{1}{r^{2+a}}\right). \tag{6.10.9}$$

But the linear terms do not make any contribution to $P^0_{(G)}$. The non-linear terms are of the order $1/r^{1+a}$ and if differentiated once

more with respect to the space coordinates will give no contribution to the surface integral. Therefore:

E. $P^0_{(G)}$ is invariant with respect to a proper time-space transformation. However, we can easily see that such a transformation does not leave the coordinate conditions invariant. Indeed we have:

$$\gamma^{*0\alpha}{}_{|\alpha} = a^0{}_{|00} + a^0{}_{|00}a^0{}_{|0} + (a^0{}_{|0}\gamma^{00})_{|0} + O\left(\frac{1}{r^{2+a}}\right), \quad (6.10.10a)$$

$$\gamma^{*mn}{}_{|n} = a^0{}_{|0m} + O\left(\frac{1}{r^{2+a}}\right). \quad (6.10.10b)$$

Here the a^0's are arbitrary, as long as the transformation is a proper one. From the above equations we conclude

$$\gamma^{*mn}{}_{|n0} - \gamma^{*0\alpha}{}_{|\alpha m} = O\left(\frac{1}{r^{2+a}}\right). \quad (6.10.11)$$

This means: we can prove that $P^0_{(G)}$ is constant under somewhat less stringent conditions than those enumerated at the beginning of this section. We can assume only the three coordinate conditions (6.10.11) instead of the four (6.10.2). The coordinate condition (6.10.11) if explicitly stated at the beginning of our argument would seem artificial. Here, however, we were led to them in a natural way by investigating the transformation properties of $P^0_{(G)}$. However, one can give a simpler and more direct proof of the following theorem:

F. We assume for $r \to \infty$:

1. $$|\gamma^{0m}| \leqslant \left| M^{0m}(t)\frac{\log r}{r^a}\right|; \quad (6.10.12)$$

2. $$\gamma^{mn}{}_{|n0} - \gamma^{0\alpha}{}_{|\alpha m} = O\left(\frac{1}{r^{2+a}}\right), \quad a > 0; \quad (6.10.13)$$

3. Λ^{0m} can be developed into a power series in x^s/r, as far as the expressions significant for the surface integral are concerned.

Then, if these conditions are fulfilled, we have:

$$P^0_{(G)} = \text{const.}$$

The proof follows along similar lines to that given in Section 9 and we shall indicate it here only briefly. We have, according to (6.7.13) and (6.7.5):

$$\dot{P}^0_{(G)} = \frac{1}{8\pi}\int_{\Sigma} \dot{K}^{m0,0\beta}{}_{|\beta} n_m\, dS = -\frac{1}{8\pi}\int_{\Sigma} \Lambda^{0m} n_m\, dS$$

$$= \int_{\Omega} \left(\dot{\mathfrak{T}}^{00} + \frac{1}{8\pi}\dot{\Lambda}^{00}\right) d\mathbf{x}. \qquad (6.10.14)$$

According to (6.9.6b) and (6.5.16) we can write the $(0, k)$ field equations in the form:

$$\tfrac{1}{2}(\gamma^{0k}{}_{|ss} - \gamma^{0\alpha}{}_{|\alpha k} + \gamma^{sk}{}_{|s0}) + \Lambda^{0k} = 0. \qquad (6.10.15)$$

But, because of coordinate condition (6.10.13) we have, identically with (6.9.7b):

$$\tfrac{1}{2}\gamma^{k0}{}_{|ss} = -\Lambda^{0k} + O\left(\frac{1}{r^{2+\alpha}}\right). \qquad (6.10.16)$$

From now on the argument is identical with that based on equation (6.9.7b) and leads to the conclusion that $\dot{P}^0_{(G)} = 0$, therefore $P^0_{(G)}$ must be constant. We can see this in a still simpler, almost trivial way. Since γ^{0k} as a solution of (6.10.16) can be developed into a power series in x^s/r, we conclude that

$$\gamma^{0m}{}_{|mk} = O\left(\frac{1}{r^{2+\alpha}}\right), \quad \alpha > 0, \qquad (6.10.17)$$

therefore our condition (6.10.13) can be written:

$$\gamma^{mn}{}_{|n0} - \gamma^{00}{}_{|0m} = O\left(\frac{1}{r^{2+\alpha}}\right). \qquad (6.10.18)$$

But a comparison with (6.8.2a) shows:

$$\gamma^{mn}{}_{|n0} - \gamma^{00}{}_{|0m} = -2\dot{K}^{m0,0\beta}{}_{|\beta}; \tag{6.10.19}$$

therefore obviously $\dot{P}^0_{(G)}$ must vanish.

The coordinate conditions (6.10.13) are, like $P^0_{(G)}$, invariant with respect to any proper space-time transformation. From equation (6.10.14) we also see that $P^0_{(G)}$ can be presented as a space integral. Obviously it does not depend on the coordinate system inside the region bounded by the infinite sphere, as long as it leads to a proper transformation on the sphere. It will also be invariant with respect to a pure space transformation which for $r \to \infty$ has the form $x^{*k} = x^k + a^k(\mathbf{x})$, where a^k is of the order zero in r, as long as its derivatives lower its order in r by one. Therefore it would seem that $P^0_{(G)}$, defined by (6.8.2a) and equal to the total gravitational mass, deserves to be regarded as the energy of the system.

It can be shown, however, that $P^0_{(G)}$ will not be constant in all coordinate systems. Indeed, let us consider a simple transformation (as an example) for $r \to \infty$:

$$x'^0 = x^0 + \varepsilon \log \frac{r}{R}. \tag{6.10.20}$$

Here ε is constant and R is the very great constant radius over which we integrate. A straightforward application of the formula (6.8.3) gives:

$$\gamma'^{00} = \gamma^{00} + \frac{\varepsilon^2}{r^2} + \varepsilon\frac{2x^s}{r^2}\gamma^{0s}, \tag{6.10.21a}$$

$$\gamma'^{0m} = \gamma^{0m} + \varepsilon\frac{x^m}{r^2} + \varepsilon\gamma^{ms}\frac{x^s}{r^2}, \tag{6.10.21b}$$

$$\gamma'^{mn} = \gamma^{mn}. \tag{6.10.21c}$$

Now to apply them to our coordinate condition, we must remember that

$$\frac{\partial}{\partial x'^\beta} = \frac{\partial}{\partial x^\beta} - \varepsilon a^\alpha{}_{|\beta}\frac{\partial}{\partial x^\alpha}. \tag{6.10.22}$$

Therefore

$$\gamma'^{00}{}_{|0m} = \gamma^{00}{}_{|0m} - \varepsilon\gamma^{00}{}_{|00}\frac{x^m}{r^2} + O\left(\frac{1}{r^{2+\alpha}}\right) \tag{6.10.23a}$$

$$\gamma'^{mn}{}_{|0n} = \gamma^{mn}{}_{|0n} - \varepsilon\gamma^{mn}{}_{|00}\frac{x^n}{r^2} + O\left(\frac{1}{r^{2+\alpha}}\right). \tag{6.10.23b}$$

The additional expressions are of the order $1/r^{1+\alpha}$ and they may change the surface integral for $\alpha = 1$. Therefore the "gravitational radiation", or rather the flow of its energy, can be created or annihilated by a choice of a coordinate system. However, as we have shown, there exist reasonable coordinate systems in which the "gravitational radiation" always vanishes.

11. RADIATION AND THE APPROXIMATION METHOD

Up to now we have shown that by a choice of the coordinate system we can annihilate the radiation. But we can also show that it can be created by a proper choice of the coordinate system and by a proper choice of the solution of the field equations.

The method used in this chapter, in all sections with the exception of the first, had little to do with the approximation procedure. We wish to come back to the ideas of the first section, applying them to General Relativity Theory. In the first section, the radiation depended on the equations of motion. Here we shall show how it depends on the particular solution we choose.

Again we shall fix our coordinate system by adding four conditions, which need to be fulfilled only at infinity. These conditions will obviously not be of the type

$$\gamma^{0\alpha}{}_{|\alpha} = 0, \quad \gamma^{mn}{}_{|n} = 0, \tag{6.11.1}$$

since they lead to no radiation. Instead we shall use de Donder's condition

$$\gamma^{\alpha\beta}{}_{|\beta} = 0 \tag{6.11.2}$$

and investigate its influence upon radiation. How does this coordinate condition change the field equations? If we apply it to (6.2.3) and (6.2.2) we obtain very simple field equations:

$$-\tfrac{1}{2}\eta^{\alpha\beta}\gamma^{\mu\nu}{}_{|\alpha\beta}+\Lambda^{\mu\nu}+8\pi\mathcal{T}^{\mu\nu}=0. \tag{6.11.3}$$

In the linear case it is the condition which leads immediately to the wave equations:

$$\Box\gamma^{\mu\nu}=0. \tag{6.11.4}$$

This is also the reason for the popularity of this coordinate condition.

Now let us try to solve the field equations at infinity with the help of our approximation method. That is, we start with the solution for big r's of

$$\Delta\underset{2}{\gamma}^{00}=0, \tag{6.11.5}$$

and according to what was said in Chapter II:

$$\underset{2}{\gamma}^{00}=4\sum_{A=1}^{N}\frac{\overset{A}{\mu}}{\overset{A}{r}}. \tag{6.11.6}$$

For $r\to\infty$ we can develop this expression, similarly as we did in Section 1:

$$\underset{2}{\gamma}^{00}=\frac{4M}{r}-4MX^{s}\left(\frac{1}{r}\right)_{|s}+2D^{sp}\left(\frac{1}{r}\right)_{|sp}+\ldots \tag{6.11.7}$$

Here

$$M=\sum_{A}\overset{A}{\underset{2}{\mu}},\qquad X^{s}=M^{-1}\sum_{A}\overset{A}{\mu}\overset{A}{\xi}{}^{s}. \tag{6.11.8}$$

$$D^{sp}=\sum_{A}\overset{A}{\mu}(\overset{A}{\xi}{}^{s}\overset{A}{\xi}{}^{p}-\tfrac{1}{3}\delta^{sp}\overset{A}{\xi}{}^{r}\overset{A}{\xi}{}^{r}). \tag{6.11.9}$$

The genesis of the last expression is simple; it springs from the following equality

$$D^{sp}\left(\frac{1}{r}\right)_{|sp}=(D^{sp}+\beta\delta^{sp})\left(\frac{1}{r}\right)_{|sp}, \tag{6.11.10}$$

where β is an arbitrary function of time. Therefore it can always be so chosen that

$$D^{ss} = 0. \tag{6.11.11}$$

Now we turn towards the field equation for $\underset{3}{\gamma}^{0m}$. We have:

$$\underset{3}{\gamma}^{0m}{}_{|ss} = 0. \tag{6.11.12}$$

Because of our coordinate conditions we then have:

$$\underset{3}{\gamma}^{0m} = 4M\dot{X}^m\frac{1}{r} - 2\dot{D}^{ms}\left(\frac{1}{r}\right)_{|s} + \beta\left(\frac{1}{r}\right)_{|m} + f^{sm}\left(\frac{1}{r}\right)_{|s}. \tag{6.11.13}$$

Here β is an arbitrary function of the time and f^{sm} an arbitrary skew-symmetric function of t. If we wish γ^{0m} to vanish at infinity, we must assume $M = \text{const}$, because of the coordinate condition $\gamma^{0\alpha}{}_{|\alpha} = 0$.

Now we can calculate γ^{mn}. Again, because of the coordinate condition, we have, if $\underset{4}{\gamma}^{mn}$ is to become Euclidean for $r \to \infty$:

$$\ddot{X}^k = 0, \qquad \dot{f}^{sm} = 0, \tag{6.11.14}$$

and

$$\underset{4}{\gamma}^{mn} = (2\ddot{D}^{mn} + \underset{4}{\dot{\beta}}\delta^{mn})\frac{1}{r} + \dots \tag{6.11.15}$$

Now if we wish to take the radiation into account we must properly choose

$$\underset{3}{\gamma}^{00}, \underset{4}{\gamma}^{0n}, \underset{5}{\gamma}^{mn}. \tag{6.11.16}$$

As was shown in Section 1 of this chapter, jumping by two steps would give only "standing waves", and therefore no solution corresponding to the retarded potential which alone gives hope for incorporating the gravitational radiation into our approximation

method. Therefore we should have to start with

$$\underset{3}{\gamma}^{00} = -4\dot{M}, \quad \underset{4}{\gamma}^{0m} = 4M\ddot{X}^{m}, \quad \underset{5}{\gamma}^{mn} = -(2\dddot{D}^{mn} + \delta^{mn}\ddot{\beta}). \quad (6.11.17)$$

Of these, only the "radiation term" in $\underset{5}{\gamma}^{mn}$ is different from zero. Let us, therefore, assume its existence and see whether it leads to a finite energy radiation. Thus assuming, according to (6.11.4),

$$X^{k} = 0, \quad f^{mn} = 0, \qquad (6.11.18)$$

we have:

$$\gamma^{00} = \frac{4M}{r} + 2D^{sp}\left(\frac{1}{r}\right)_{|sp} + \ldots,$$

$$\gamma^{0m} = -2\dot{D}^{ms}\left(\frac{1}{r}\right)_{|s} + \beta\left(\frac{1}{r}\right)_{|m} + \ldots, \qquad (6.11.19)$$

$$\gamma^{mn} = (2\ddot{D}^{mn} + \delta^{mn}\dot{\beta})\frac{1}{r} - 2\dddot{D}^{mn} - \delta^{mn}\ddot{\beta}.$$

To find the energy radiation we shall now use formula (6.7.13):

$$8\pi\dot{P}^{0}_{(G)} = \int_{\Sigma} \dot{K}^{m0,0\beta}{}_{|\beta} n_m dS = -\int_{\Sigma} \Lambda^{0m} n_m dS \qquad (6.11.20)$$

which in our coordinate system, because of (6.6.15), gives:

$$-8\pi\dot{P}^{0}_{(G)} = \int_{\Sigma} (\gamma^{00}{}_{|0m} + \gamma^{0m}{}_{|00})\, n_m = \int_{\Sigma} \Lambda^{0m} n_m dS. \quad (6.11.21)$$

We may expect the radiation terms to be different from zero for the even orders of Λ^{0m}. The first order in which this can happen is for $\underset{8}{\Lambda}^{0m}$. Indeed, without going into detail, we see that such a contribution may come from the expression in Λ^{0m} which is:

$$\sim \underset{5}{\gamma}^{mn}{}_{|0}\,\underset{12}{\gamma}^{00}{}_{|n}. \qquad (6.11.22)$$

Such an expression, because $D^{ss} = 0$, will be proportional to

$$\dddot{\underset{6}{\beta}}\,\underset{2}{\dot{M}}. \tag{6.11.23}$$

Therefore, it is generally possible to obtain radiation of the eighth order, but only if for $r \to \infty$ we have solutions of the kind (6.11.19) with $\dddot{\underset{6}{\beta}} \neq 0$. Otherwise, if we take a different world, with $\beta = 0$ we do not have any radiation of the eighth order. Thus the radiation can be annihilated or created by the proper choice of solutions. The simplest solution with $\beta = 0$ leads to no radiation. If we take $\beta \neq 0$ then we have radiation! From the left-hand side of equation (6.11.21), that is from the linear part of the surface integrals, we see that they must be equal to the right-hand side already calculated. This means that we must assume

$$\underset{6}{\gamma}{}^{0m}_{|} = C(t)\left(\frac{1}{r}\right)_{|m}, \tag{6.11.24}$$

so that the integral of the left-hand side will be exactly equal to the contribution (6.11.23). If $\beta = 0$, then of course $C = 0$ too.

Now let us assume that we have chosen $\beta = 0$ in (6.11.19). Then to find the radiation we must proceed one step further, calculating Λ^{0m} of the tenth order. For example, there will be expressions of the type

$$\underset{7}{\gamma}{}^{mn}{}_{|s}\,\underset{3}{\dot{\gamma}}{}^{0n}{}_{|s}. \tag{6.11.25}$$

Now

$$\underset{7}{\gamma}{}^{mn} \sim r^2 \frac{d^5}{dt^5} D^{mn}. \tag{6.11.26}$$

That is: $\underset{7}{\gamma}{}^{mn}$ will have an expression of this type generated in the next approximation by $\dddot{\underset{5}{D}}{}^{sp}$. On the other hand, $\underset{3}{\gamma}{}^{0n}$ has an expression of the type

$$\dot{D}^{ms}\left(\frac{1}{r}\right)_{|s}.$$

Therefore the product (6.11.25) will be of the order $1/r^2$ and will give a contribution to the surface integral. It will be of the right order and of the form:

$$\dot{D}^{mn} \frac{d^5}{dt^5} D^{mn}. \tag{6.11.27}$$

Generally we shall have many expressions of a similar form; this is

$$\underset{10}{A}^{0m} \sim \underset{10}{f}(D^2). \tag{6.11.28}$$

It would seem, therefore, that a radiation must appear in the tenth order. However, this is not so. At each approximation step we may add an arbitrary function of time. Therefore, let us say that in the 7th order we may add to γ^{mn} an arbitrary function of time of the form

$$\underset{7}{\beta}(t)\,\delta^{mn}. \tag{6.11.29}$$

As we have shown before, such an added expression will give a contribution of the form

$$\underset{8}{\dot{\beta}}\underset{2}{M}. \tag{6.11.30}$$

Therefore, again, by a proper choice of an additive harmonic function of the seventh order we can annihilate the radiation created by an additive harmonic function of the fifth order! We may add that such a harmonic function does not upset the accepted coordinate conditions.

What is the moral of this chapter? The results are indeed meagre and mostly of a negative character. They show that it is hardly possible to connect any physical meaning with the flux of energy and momentum tensor defined with the help of the pseudo-energy-momentum tensor. Indeed, the radiation can be annihilated by a proper choice of the coordinate system. On the other hand, if we use a coordinate system in which the flux of energy may exist, then it can be made whatever we like by the addition of proper harmonic functions starting with $\underset{5}{\gamma}^{mn}$ as a function of time.

In the linear theory we were faced with the choice between the retarded and advanced potential. Here in the theory of gravitation the choice is not so simple. Using the approximation procedure, we are faced with the choice between single and double jumps. We can speak only about radiation in the case of single jumps. However, its existence or non-existence or its value will depend upon the choice of arbitrary harmonic functions.

Yet the full equation (6.5.1) makes sense, though not if we treat each of the sides as the definition of the energy-momentum flux. It makes sense since it is a consequence of the equations of motion. Indeed, the surface integrals taken around the singularities give us the equations of motion. Taken over the surface surrounding all singularities, they give us conservation laws which are a consequence of the equations of motion.

Einstein often remarked: "We do not have any satisfactory classical theory of radiation. Ritz understood this fact. He was an intelligent man ...". This remark seems especially apt if applied to gravitational radiation.

APPENDICES

1. THE δ FUNCTION

In this work we use mostly Dirac's δ functions of a special kind; therefore we shall discuss their structure in detail.

The ordinary δ functions may be introduced into the theory in various ways. We shall mention three of them:

(A): the axiomatic method,

(B): the method of Fourier transforms,

(C): the realistic method, treating the δ functions as a limiting case of spread- out sources.

We shall begin with the one-dimensional δ function. The generalization to a δ function of more dimensions is almost trivial.

(A): In the axiomatic method we stipulate for $\delta(x)$ the following properties:

(A_1): one can formally differentiate $\delta(x)$ as many times as one wishes,

(A_2): for $x \neq 0$ we have $\delta(x) = 0$; for $x = 0$, we have $\delta(0) = \infty$,

(A_3): for an arbitrary neighbourhood $V(x_0)$ of x_0 and for an arbitrary function $f(x)$ continuous at x_0 we have:

$$\int_{V(x_0)} dx\,\delta(x-x_0)f(x) = \int_{-\infty}^{\infty} dx\,\delta(x-x_0)f(x) = f(x_0). \qquad \text{(A.1.1)}$$

(B): In the method of Fourier transforms, the symbol $\delta(x'-x'')$ is defined by the following equation:

$$\delta(x'-x'') = \sum_{\lambda} \psi_{\lambda}^{*}(x')\psi_{\lambda}(x''). \qquad \text{(A.1.2)}$$

Here the $\psi_\lambda(x)$ are the eigenfunctions of an Hermitian operator Λ labelled by the eigenvalues λ.

We assume that differentiation and integration are commutative with summation over the spectrum. From this and from the properties of a complete set of eigenfunctions one "proves" the validity of (A_1—A_3). Of special importance is the definition of $\delta(x)$ through the eigenfunctions of the operator id/dx, that is $(2\pi)^{-1/2}e^{-ikx}$ which according to (A.1.2) gives:

$$\delta(x'-x'') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(x'-x'')}. \tag{A.1.3}$$

Such an "ordinary" Fourier representation of the δ function plays an especially important role in calculations referring to Green's functions.

(C): We shall discuss the third, realistic method. In it we consider $\delta(x)$ as a limit as $\varepsilon \to 0$, of certain ordinary functions $\delta(\varepsilon, x)$, with the following properties:

(C_1): $\delta(\varepsilon, x)$ have all derivatives as functions of x and ε,

(C_2): $\lim\limits_{\varepsilon\to 0} \delta(\varepsilon, x) = 0$ for $x \neq 0$,

(C_3): For an arbitrary neighbourhood $V(x_0)$ of x_0 and for an arbitrary continuous function $f(x)$ in $V(x_0)$, there exists:

$$\lim_{\varepsilon\to 0} \int_{V(x_0)} dx\, \delta(\varepsilon, x-x_0) f(x) = f(x_0). \tag{A.1.4}$$

Indeed, by going to the limit $\varepsilon \to 0$ and interchanging the limiting process with the integration in (C_3) we justify the axiomatic properties (A_1-A_3).

(Sometimes we have to strengthen our assumptions by stipulating how strongly $\delta(\varepsilon, x)$ goes to zero with increasing x.)

The realistic method boils down to the following procedure: all calculations have to be performed, not on $\delta(x)$ but on $\delta(\varepsilon, x)$. The transition to the limit $\varepsilon \to 0$ has to be made in the last result. Only if such a result does not depend on the special form of $\delta(\varepsilon, x)$, but is determined by its satisfying (C_1-C_3), can we expect the same

result when operating, not with $\delta(\varepsilon, x)$, but with a $\delta(x)$ satisfying (A_1-A_3). In other words, the realistic method can be regarded — at least by the physicists — as a justification for the axiomatic method. This justification lies essentially in the fact that the class of functions $\delta(\varepsilon, x)$ satisfying (C_1-C_3) is not empty.

One of the possible examples of functions belonging to such a class (perhaps the simplest one) is:

$$\delta(\varepsilon, x) = \frac{1}{\sqrt{2\pi}\,\varepsilon}\, e^{-x^2/2\varepsilon^2}. \tag{A.1.5}$$

None of these three methods can be regarded as satisfactory from the mathematical point of view. However, generally speaking they are sufficient for the application of the δ functions in theoretical physics. Even this statement is not always true. They are, of course, sufficient if in the special calculations only products of δ functions with continuous functions appear. However, in general, and in particular here, in the problem of motion in G. R. T., we shall have to deal with products of δ's with functions singular at precisely the point at which the δ functions are infinite. Hence we are faced with the problem of interpreting the expression:

$$\int_{-\infty}^{\infty} dx \frac{\delta(x)}{|x|^p}, \quad p > 0. \tag{A.1.6}$$

Such an integral is usually considered as divergent. It is this fact which produces many difficulties.

What are the logical foundations of the statement that the last expression is divergent for a certain $p > 0$? Usually one appeals to (A_3). From this it appears to follow that the use of δ substitutes 0 for the argument x in $1/|x|^p$. Hence, we conclude that the integral (A.1.6) must be infinite.

This result will seem less well-founded if we consider that A_3 is valid only for functions continuous at least at $x = x_0$. Thus the last expression is neither divergent nor convergent. It is simply meaningless.

Indeed we can show that we can consistently introduce modified Dirac δ functions, which we shall call $\hat{\delta}$ and $\hat{\hat{\delta}}$, such that for any chosen $p > 0$ we have either

$$\int_{-\infty}^{\infty} dx \frac{\hat{\delta}(x)}{|x|^p} = 0, \tag{A.1.7}$$

or

$$\int_{-\infty}^{\infty} dx \frac{\hat{\hat{\delta}}(x)}{|x|^p} = \omega_p,$$

where ω_p is any arbitrarily assigned value. Obviously $\hat{\delta}$ as well as $\hat{\hat{\delta}}$ will depend on the particular p chosen.

Thus we shall define $\hat{\hat{\delta}}$ by the following set of axioms:

($\hat{\hat{A}}_1$) $\hat{\hat{\delta}}$ has all derivatives for $x \neq 0$.

($\hat{\hat{A}}_2$) $\hat{\hat{\delta}}(x) = 0$ if $x \neq 0$. $\hat{\hat{\delta}}(0)$ is undefined. $\hat{\hat{\delta}}(x)$ under the integral should be treated as an ordinary function.

($\hat{\hat{A}}_3$) For every continuous function $f(x)$ and for an arbitrary neighbourhood $V(x_0)$ of x_0 we have

$$\int_{V(x_0)} dx\, \hat{\hat{\delta}}(x - x_0) f(x) = f(x_0). \tag{A.1.8}$$

($\hat{\hat{A}}_4$) For a certain p we have

$$\int_{V(0)} dx \frac{\hat{\hat{\delta}}(x)}{|x|^p} = \omega_p \tag{A.1.9}$$

where ω_p is a previously assigned value.

Let us start with the realistic introduction of the $\hat{\delta}$ and $\hat{\hat{\delta}}$ which, as has been said before, can be regarded as a justification for their axiomatic introduction. We start from a realistic representation of the ordinary Dirac functions $\delta(\varepsilon, x)$ — that is, such that satisfy

(C_1-C_3). Now we define $\hat{\delta}(\varepsilon, x)$ and $\hat{\hat{\delta}}(\varepsilon, x)$ in the following simple manner:

$$\hat{\delta}(\varepsilon, x) = a(\varepsilon)\,|x|^p \frac{d}{dx}\big(x\delta(\varepsilon, x)\big), \tag{A.1.10}$$

$$\hat{\hat{\delta}}(\varepsilon, x) = \hat{\delta}(\varepsilon, x)(1+\omega_p|x|^p). \tag{A.1.11}$$

First, let us show that $\hat{\delta}(\varepsilon, x)$ satisfies the axioms $(\hat{A}_1\text{-}\hat{A}_4)$ in the limit $\varepsilon \to 0$. This is trivial for $\hat{A}_1$ and $\hat{A}_2$. In order to satisfy $(\hat{A}_3)$ we have to choose a in (A.1.10) so that

$$\int_{-\infty}^{\infty} \hat{\delta}(\varepsilon, x)\,dx = 1 \tag{A.1.12}$$

is satisfied. This is certainly always possible and a turns out to be $\sim \varepsilon^{-p}$. (For example for $\delta(\varepsilon, x) = (2\pi\varepsilon^2)^{-1/2}e^{-x^2/2\varepsilon^2}$ we find $a = \varepsilon^{-p}\sqrt{\pi}\left[2^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)\right]^{-1}$.) We can also easily see that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \hat{\delta}(\varepsilon, x)\,|x|^n\,dx = 0 \quad \text{for} \quad n > 0. \tag{A.1.13}$$

From the last two equations and from the fact that $\delta(\varepsilon, x)$ vanishes quickly enough for $x \neq 0$, $\hat{A}_3$ follows, precisely as for ordinary Dirac functions.

Now, $\hat{A}_4$ follows from:

$$\int_{-\infty}^{\infty} \frac{\hat{\delta}(\varepsilon, x)}{|x|^p}\,dx = a(\varepsilon)\,\delta(\varepsilon, x)x\Big|_{-\infty}^{\infty} = 0. \tag{A.1.14}$$

Therefore the axioms $(\hat{A}_1-\hat{A}_4)$ are justified. Similarly we can find for $\hat{\hat{\delta}}(x)$ that

$$\int_{-\infty}^{\infty} \hat{\hat{\delta}}(\varepsilon, x)\,dx = 1, \qquad \int_{-\infty}^{\infty} \frac{\hat{\hat{\delta}}(\varepsilon, x)}{|x|^p}\,dx = \omega_p. \tag{A.1.15}$$

However, we can now go further. We can disregard any realistic representation, and thus write formally the following symbolic equations:

$$\left.\begin{aligned} \hat{\delta}(x) &= \alpha|x|^p \frac{d}{dx}\big(x\delta(x)\big), \\ \hat{\hat{\delta}}(x) &= \hat{\delta}(x)(1+\omega_p|x|^p) \end{aligned}\right\} \tag{A.1.16}$$

where the $\delta(x)$ are ordinary Dirac functions and the α's infinite constants so chosen that

$$\int_{V(0)} \hat{\delta}\,dx = 1 = \int_{V(0)} \hat{\hat{\delta}}\,dx.$$

The generalization to more dimensions is simple enough. Let us assume a two-dimensional manifold (x, y) and a function $f(x, y)$, so that $f(x, y) \to 0$ as $x, y \to 0$. We wish to find such $\hat{\delta}(x, y)$, that

$$\int_{V(0)} \hat{\delta}(x, y)\,dx\,dy = 1 \tag{A.1.17}$$

and

$$\int_{V(0)} \hat{\delta}(x, y)|f|^{-1}\,dx\,dy = 0. \tag{A.1.18}$$

The answer is symbolically written:

$$\hat{\delta}(x, y) = \alpha|f(x, y)|\frac{\partial^2}{\partial x\,\partial y}[xy\,\delta(x, y)] \tag{A.1.19}$$

assuming, of course, that $|f|$ is such that the integral (A.1.17) does not become identically zero for an arbitrary α.

From the point of view of our applications the most important case is the three-dimensional one in which the δ's, the f's and therefore also the $\hat{\delta}$ and $\hat{\hat{\delta}}$ depend on r only, where $r^2 = x^2+y^2+z^2$. Such central symmetrical cases can be reduced to one with only one dimension.

For an ordinary three-dimensional Dirac $\delta(\mathbf{x})$ function we have:

$$\int d\mathbf{x}\,\delta(\mathbf{x}) = 1.$$

We can satisfy the last equation, as well as other conditions which should be satisfied by the $\delta(\mathbf{x})$ function, putting

$$\delta(\mathbf{x}) \to \frac{1}{2\pi r^2}\,\delta(r) \quad \text{where} \quad \int\limits_0^\infty \delta(r)\,dr = \tfrac{1}{2}. \tag{A.1.20}$$

However, for $\hat{\delta}(r)$ and $\hat{\hat{\delta}}(r)$, besides the normalization conditions identical with those above, we also wish the following equations to be satisfied:

$$\int\limits_0^\infty \frac{\hat{\delta}(r)}{r^p}\,dr = 0 \quad \text{and} \quad \int\limits_0^\infty \frac{\hat{\hat{\delta}}(r)}{r^p}\,dr = \tfrac{1}{2}\omega_p. \tag{A.1.21}$$

Thus if we define:

$$\hat{\delta}(\mathbf{x}) \to \frac{1}{2\pi r^2}\,\hat{\delta}(r), \qquad \hat{\hat{\delta}}(\mathbf{x}) \to \frac{1}{2\pi r^2}\,\hat{\hat{\delta}}(r) \tag{A.1.22}$$

we see that

$$\int d\mathbf{x}\frac{\hat{\delta}(\mathbf{x})}{|\mathbf{x}|^p} = 0, \qquad \int d\mathbf{x}\frac{\hat{\hat{\delta}}(\mathbf{x})}{|\mathbf{x}|^p} = \omega_p. \tag{A.1.23}$$

These $\hat{\delta}$ (or $\hat{\hat{\delta}}$) functions behave as kernels of integral operations with respect to continuous functions exactly like ordinary Dirac δ functions. They allow us, however, to connect definite meanings with integrals of products of these $\hat{\delta}$ or $\hat{\hat{\delta}}$ functions with those that become infinite for $\mathbf{x} \to 0$. From our point of view, this is especially important for the $\hat{\delta}$ functions. We shall assume that they are so chosen that the integrals over these $\hat{\delta}$ functions times functions that become infinite at $\mathbf{x} = 0$, vanish in each particular case. We shall call such properly chosen $\hat{\delta}$ functions the "good" δ functions. The application of these functions is equivalent to the regularization

procedure, which in this way is incorporated into the mathematical tool which we are using.

Sometimes we look for good three-dimensional δ functions which satisfy our condition (A.1.23), not only for a certain p but for a finite sequence of the p's, say $p = 1, 2, 3, \ldots, k$. We shall show that we can just as well construct a realistic model of such a function. Let us start with a model of an ordinary Dirac δfunction satisfying the following condition:

$$\delta(\varepsilon, \mathbf{x}) = \varepsilon^{-3} \Delta\left(\frac{|\mathbf{x}|}{\varepsilon}\right) \tag{A.1.24}$$

where $\Delta(z)$ is such that

$$D_p = 4\pi \int_0^\infty dz\, z^{2-p} \Delta(z) \equiv \int_\infty d\mathbf{z}\, \Delta(|\mathbf{z}|)\, |\mathbf{z}|^{-p} \tag{A.1.25a}$$

with

$$D_0 = 1, \qquad D_p \neq 0, \qquad (p = 1, 2, \ldots, k). \tag{A.1.25b}$$

Furthermore, we assume that $\Delta(z)$, defined in the interval $(0, \infty)$, has all its derivatives and for $z \to \infty$ they reduce to zero at least exponentially.

For practically every model of an ordinary Dirac δ function, we can find a model of a Δ function satisfying (A.1.25). If the chosen function $\Delta(z)$ does not have this property, it can be made to have it by multiplying it by z^{k-2} and renormalizing. For example, if

$$\delta(\varepsilon, \mathbf{x}) = (2\pi)^{-3/2} \varepsilon^{-3} e^{-1/2|\mathbf{x}|^2\varepsilon^{-2}},$$

or

$$\Delta(z) = (2\pi)^{-3/2} e^{-1/2z^2}$$

does not have this property, then

$$\Delta(z) = \frac{1}{2\pi \cdot 2^{(k+1)/2}\, \Gamma\left(\dfrac{k+1}{2}\right)} z^{k-2}\, e^{-1/2z^2}$$

does. In this case the D_p's are:

$$D_p = 2^{-p/2}\,\Gamma\left(\frac{k-p+1}{2}\right)\Big/\Gamma\left(\frac{k+1}{2}\right).$$

Now that we have the $\Delta(z)$ satisfying (A.1.25), and therefore the model of an ordinary Dirac δ function, we canf ind a simple example of the good $\hat{\delta}(\varepsilon, \mathbf{x})$ function. This is:

$$\hat{\delta}(\varepsilon, \mathbf{x}) = \hat{T}_\varepsilon \delta(\varepsilon, \mathbf{x}), \tag{A.1.26}$$

where $\hat{T}_\varepsilon$ is the simple operator

$$\hat{T}_\varepsilon = \frac{1}{k!}\left(\frac{\partial}{\partial\varepsilon}\right)^k \varepsilon^k. \tag{A.1.27}$$

Indeed, we then have:

$$\int_\infty \hat{\delta}(\varepsilon, \mathbf{x})\,|\mathbf{x}|^{-p}\,d\mathbf{x} = \frac{1}{k!}\left(\frac{\partial}{\partial\varepsilon}\right)^k \varepsilon^{k-p}\int_0^\infty \Delta(z) z^{2-p}\,dz$$

$$= \frac{1}{k!}\left(\frac{\partial}{\partial\varepsilon}\right)^k \varepsilon^{k-p} D_p = \begin{cases} 1 & \text{for} \quad p = 0, \\ 0 & \text{for} \quad p = 1, 2, \ldots, k. \end{cases} \tag{A.1.28}$$

It is slightly more complicated to define $\hat{\hat{\delta}}(\varepsilon, \mathbf{x})$. As before, we first introduce a δ function which satisfies (A.1.25). Then we define:

$$\hat{\hat{\delta}}(\varepsilon, \mathbf{x}) = \hat{\hat{T}}_\varepsilon \delta(\varepsilon, \mathbf{x}), \tag{A.1.29}$$

$$\hat{\hat{T}}_\varepsilon = \frac{1}{k!}\sum_{s=0}^{k}(-1)^s \frac{\omega_s}{D_s}\binom{k}{s}\left(\frac{\partial}{\partial\varepsilon}\right)^{k-s}\varepsilon^{k-s}\left(\varepsilon^2\frac{\partial}{\partial\varepsilon}\right)^s. \tag{A.1.30}$$

First we can see that $\hat{\hat{T}}_\varepsilon$ is a generalization of $\hat{T}_\varepsilon$ and goes over into $\hat{T}_\varepsilon$ for $\omega_p = 0$, $p = 1, 2, \ldots, k$.

We shall now calculate the integral

$$\int_\infty d\mathbf{x}\,\hat{\hat{\delta}}(\varepsilon, \mathbf{x})\,|\mathbf{x}|^{-p} = \hat{\hat{T}}_\varepsilon \varepsilon^{-p} 4\pi \int_0^\infty dz\, z^{2-p}\Delta(z) = D_p \hat{\hat{T}}_\varepsilon \varepsilon^{-p}. \tag{A.1.31}$$

We know that

$$(-1)^s\left(\varepsilon^2\frac{\partial}{\partial\varepsilon}\right)^s\varepsilon^{-p}=\left(\frac{\partial}{\partial\varepsilon^{-1}}\right)^s(\varepsilon^{-1})^p=\begin{cases}0 & \text{for} \quad s>p,\\ s!\binom{p}{s}\varepsilon^{s-p} & \text{for} \quad s\leqslant p.\end{cases} \tag{A.1.32}$$

Therefore

$$\hat{\hat{T}}_s\varepsilon^{-p}=\frac{1}{k!}\sum_{s=0}^{p}\frac{\omega_s}{D_s}s!\binom{k}{s}\binom{p}{s}\left(\frac{\partial}{\partial\varepsilon}\right)^{k-s}\varepsilon^{k-p}. \tag{A.1.33}$$

We also know that

$$\left(\frac{\partial}{\partial\varepsilon}\right)^{k-s}\varepsilon^{k-p}=\begin{cases}0 & \text{for} \quad s<p,\\ (k-p)! & \text{for} \quad s=p.\end{cases} \tag{A.1.34}$$

Therefore, finally

$$\begin{aligned}\int_\infty d\mathbf{x}\hat{\hat{\delta}}(\varepsilon,\mathbf{x})|\mathbf{x}|^{-p}&=D_p\hat{\hat{T}}_s\varepsilon^{-p}=D_p\frac{\omega_p}{D_p}\binom{k}{p}\binom{p}{p}\frac{p!(k-p)!}{k!}\\&=\begin{cases}1 & \text{for} \quad p=0,\\ \omega_p & \text{for} \quad p=1,2,\ldots,k.\end{cases}\end{aligned} \tag{A.1.35}$$

There is one more formal problem that we must discuss here and which is connected with the use of $\hat{\delta}$ (or $\hat{\hat{\delta}}$) functions. We shall start our discussion by an example. Let us assume that in some calculations, operating with the (one dimensional) $\hat{\delta}$ functions we came across an expression

$$\int_\infty\frac{\hat{\delta}(x')\,\hat{\delta}(x'')}{|x'-x''|}dx'\,dx''. \tag{A.1.36}$$

We ask: what is the meaning of such an expression? In its place, let us write:

$$\lim_{\varepsilon\to0}\int_\infty\frac{\hat{\delta}(x')\,\hat{\delta}(\varepsilon,x'')}{|x'-x''|}dx'\,dx''. \tag{A.1.37}$$

But the expression in brackets is well-defined and, according to axiom $(\hat{A}_3)$, is equal to

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{\hat{\delta}(\varepsilon, x'')}{|x''|}\,dx'' = \int_{-\infty}^{\infty}\frac{\hat{\delta}(x'')}{|x''|}\,dx''. \tag{A.1.38}$$

This expression, however, vanishes for a properly chosen $\hat{\delta}$. We also see that the result does not depend on whether we spread out $\hat{\delta}(x')$ or $\hat{\delta}(x'')$. This is so because the function $|x'-x''|^{-1}$ is symmetrical in x', x''.

Obviously we can generalize this theorem for an integral over a product of an arbitrary number of one-dimensional $\hat{\delta}$'s. The integral is defined if one of the $\hat{\delta}$'s is the $\hat{\delta}$ function and the rest of them spread out $\hat{\delta}(\varepsilon, x)$ functions with the same ε. Thus we have the following theorem:

$$\int_{\infty} f(x_1, \ldots, x_n)\,\hat{\delta}(x_1)\ldots\hat{\delta}(x_n)\,dx_1\ldots dx_n = f(0, \ldots, 0) \tag{A.1.39}$$

if $f(0, \ldots, 0)$ is finite, and vanishes if $f(0, \ldots, 0) = \infty$ and if the $\hat{\delta}$ can be and are so chosen that $\int_{\infty}\hat{\delta}(x_k)f(0, \ldots, x_k, \ldots 0)\,dx_k$ vanishes for every k. Further generalizations of this procedure for a, say, twice-taken three-dimensional manifold are trivial. The rule formulated here will allow us to avoid all indefinite expressions appearing in our calculations.

"Mass points", "charge points", are, of course, fictional concepts introduced into the theory in order to describe simply the complicated reality. It is just these concepts that are adequately described by Dirac's δfunction.

On the other hand, if we have a continuous distribution of density, say, concentrated around a point x_0, then the space integrals of the products of ϱ times $|\mathbf{x}-\mathbf{x}_0|^{-p}$ will generally exist and have definite values. This is obviously true it $\varrho(\mathbf{x}_0) \neq 0$ and $p < 3$. The values of such integrals, corresponding to our ω_p's

characterize to a certain extent the continuous structure of the density function.

Thus our $\hat{\hat{\delta}}$ reduces such a continuous distribution to a point in such a way as to leave in our calculation a skeleton of the internal structure of the "mass points" or "charge points" in the form of ω_p. However, this can always be wiped out by the renormalization process. Therefore, it is more convenient always to use the $\hat{\delta}$ instead of the δ, or instead of the $\hat{\hat{\delta}}$'s to avoid both the infinities and the renormalization procedure.

To conclude: in this book we use this convenient tool of reasoning, that is the $\hat{\delta}$, corresponding to the choice $\omega_p = 0$. If nothing is said to the contrary, our δ's are the good $\hat{\delta}$'s, denoted in this section by $\hat{\delta}$.

2. THE FIELD VALUES ON THE WORLD-LINES

The solutions of field equations which we meet in this book are functions of world points $x^\alpha = (x^0, x^k)$; but they depend functionally on motions of singularities.

Let us assume that we have N singularities and that their motions are given parametrically by $\overset{A}{\xi}{}^\alpha = \overset{A}{\xi}{}^\alpha(\overset{A}{\lambda})$. Here the index A labels the singularity (and so $A = 1, 2, \ldots N$). The summation over A is to be performed only if explicitly stated. By $\overset{A}{\lambda}$ we denote the invariant parameter of the A'th world-line. Therefore, if we denote a component of the field by f, we have:

$$f = f(x^\alpha)[\overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{N}{\xi}], \tag{A.2.1}$$

where the square bracket is reserved for functional dependence.

Let us now consider the equation:

$$x^0 = \overset{A}{\xi}{}^0(\overset{A}{\lambda})$$

and assume that it determines $\overset{A}{\lambda}$ uniquely as a function of x^0: $\overset{A}{\lambda} = \overset{A}{\lambda}(x^0)$. Then the equations of the A'th world-line can be expressed in terms of the parameter x^0:

$$\overset{A}{\xi}{}^a = \overset{A}{\xi}{}^a(x^0); \quad a = 1, 2, 3; \quad A = 1, 2, \ldots, N.$$

These functions, if analytic on the x^0 axis, are fully determined by their values and those of all their derivatives at one arbitrary point. Therefore, the functional dependence can be changed into ordinary dependence on an infinite set of arguments, that is, on all the derivatives at a certain moment $\overset{A}{x}{}^0$. The $\overset{A}{x}{}^0$ can be so chosen as to be the same for all particles and the same as the x^0 appearing in $f(x^0, x^k)$. Therefore, we can write the f of (A.2.1) in the form:

$$f = f(x^0, x^a, \overset{A}{\xi}{}^a(x^0), \overset{A}{\xi}{}^a(x^0)_{|0}, \overset{A}{\xi}{}^a(x^0)_{|00}, \ldots). \tag{A.2.2}$$

Because of the approximation method used, in the actual applications we shall have only time derivatives of the first and second order in expressions of the above type.

The function f may (and usually will) have a singularity on the A'th world-line. Yet, in spite of this, we can connect a certain definite meaning with the value of f along the A'th world-line. To do this we shall use our good δ's. We define f along the world-line A, denoted by $\overset{A}{\tilde{f}}$, by means of the equation:

$$\overset{A}{\tilde{f}} = \int d\mathbf{x}\, \delta\big(\mathbf{x} - \overset{A}{\boldsymbol{\xi}}(x^0)\big)\, f(x^0, \mathbf{x})\,[\overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{N}{\xi}]. \tag{A.2.3}$$

Without much loss of generality, we now assume that there is only one world-line, and we drop the writing of "1" above the $\sim$.

We ask: what is the meaning of defining f along the world-line, or, as we shall say for short, of the "tweedling" process?

We start with a simple example:

$$f = \frac{a_{-1}}{|\mathbf{x} - \boldsymbol{\xi}|} + a_0 + a_s(x^s - \xi^s) + \tfrac{1}{2}a_{sr}(x^s - \xi^s)(x^r - \xi^r) + \ldots \tag{A.2.4}$$

where the a's are some functions of x^0 only. Then using our definition (A.2.3), we have:

$$\tilde{f} = a_0, \quad \widetilde{f_{|s}} = a_s, \quad \widetilde{f_{|sr}} = a_{sr}. \tag{A.2.5}$$

Thus the process of tweedling consists in two things: first: in ignoring the singular part of f; second: in introducing the expressions $\xi^k(x^0)$ instead of x^k into the regular part of f. In this way the tweedled expression becomes a function of x^0 only.

Let us now assume that f is a sum of expressions of the following type:

$$\varphi = \frac{1}{|\mathbf{x}-\boldsymbol{\xi}|^n} \frac{(x^1-\xi^1)^p(x^2-\xi^2)^q(x^3-\xi^3)^s}{|\mathbf{x}-\boldsymbol{\xi}|^{p+q+s}}. \tag{A.2.6}$$

Then we see that

1°: $\tilde{\varphi} = 0$ for $n > 0$;

2°: $\tilde{\varphi} = 0$ if any of p, q, s is odd. This is true for arbitrary n, therefore also for $n = 0$, because $\delta(\mathbf{x})$ is spherically symmetric in $\mathbf{x}$;

3°: If $n = 0$ and p, q, s are even, $\tilde{\varphi}$ can easily be calculated by the use of a polar coordinate system.

In making use of the tweedling operation prescribed by (A.2.3) we must be careful to distinguish between

$$\widetilde{\varphi_{|s}} \quad \text{and} \quad \frac{\partial \tilde{\varphi}}{\partial \xi^s} = \tilde{\varphi}_{|\xi^s} \tag{A.2.7}$$

which we shall also occasionally denote by $\tilde{\varphi}_{|s}$.

These are

$$\widetilde{\varphi_{|s}} = \int \varphi_{|s}\, \delta(\mathbf{x}-\boldsymbol{\xi})\, d\mathbf{x} \tag{A.2.8}$$

$$\tilde{\varphi}_{|s} = \frac{\partial}{\partial \xi^s} \int \varphi\, \delta(\mathbf{x}-\boldsymbol{\xi})\, d\mathbf{x} = \frac{\widetilde{\partial \varphi}}{\partial \xi^s} + \widetilde{\varphi_{|s}}. \tag{A.2.9}$$

Therefore $\breve{\varphi}_{|s}$ and $\tilde{\varphi}_{|s}$ are equal only if

$$\frac{\breve{\partial\varphi}}{\partial\xi^s} = \breve{\varphi}_{|\xi^s} = 0, \tag{A.2.10}$$

that is if φ does not depend explicitly on ξ^s. Otherwise, we have, in general:

$$\tilde{\varphi}_{|s} = \frac{\partial\tilde{\varphi}}{\partial\xi^s} = \frac{\breve{\partial\varphi}}{\partial\xi^s} + \breve{\varphi}_{|s}. \tag{A.2.11}$$

One more formula which will play an important role later and which follows from the definition (A.2.3) is:

$$\tilde{\varphi}_{|0} = \frac{d\tilde{\varphi}}{dx^0} = \frac{d}{dx^0}\int \varphi\,\delta(\mathbf{x}-\boldsymbol{\xi})\,d\mathbf{x} = \breve{\varphi}_{|0} + \breve{\varphi}_{|s}\,\xi^s{}_{|0} \tag{A.2.12}$$

and because $\xi^0{}_{|0} = x^0{}_{|0} = 1$, we have:

$$\tilde{\varphi}_{|0} = \breve{\varphi}_{|\alpha}\,\xi^\alpha{}_{|0} = \breve{\varphi}_{|0} + \breve{\varphi}_{|s}\,\xi^s{}_{|0}. \tag{A.2.13}$$

Let us now assume the existence of two field quantities ψ and φ, both singular at the world-line ξ. We ask: is it true that

$$\breve{\varphi\psi} = \tilde{\varphi}\tilde{\psi}? \tag{A.2.14}$$

This is certainly so if neither of the fields is singular. But if at least one of them is singular this does not need to be true. Indeed, the singular part of, say, ψ, multiplied by the regular part of φ, may give a contribution to $\breve{\psi\varphi}$ not present in $\tilde{\psi}\tilde{\varphi}$, in which only the regular parts of both functions appear. But there is one special case, important in later applications, in which (A.2.14) is satisfied. This is the case in which the singular parts of ψ and φ involve only odd powers of $|\mathbf{x}-\boldsymbol{\xi}|$, e. g.:

$$\begin{aligned}
\psi &= \frac{a_{-(2s+1)}}{|\mathbf{x}-\boldsymbol{\xi}|^{2s+1}} + \ldots + \frac{a_{-3}}{|\mathbf{x}-\boldsymbol{\xi}|^3} + \frac{a_{-1}}{|\mathbf{x}-\boldsymbol{\xi}|} + \text{regular part} \\
\varphi &= \frac{b_{-(2p+1)}}{|\mathbf{x}-\boldsymbol{\xi}|^{2p+1}} + \ldots + \frac{b_{-3}}{|\mathbf{x}-\boldsymbol{\xi}|^3} + \frac{b_{-1}}{|\mathbf{x}-\boldsymbol{\xi}|} + \text{regular part.}
\end{aligned} \tag{A.2.15}$$

We see, in this case, that the only possible contributions coming from the products of singular and regular expressions are of the type:

$$\frac{(x^1-\xi^1)^s(x^2-\xi^2)^p(x^3-\xi^3)^q}{|\mathbf{x}-\boldsymbol{\xi}|^{2n+1}}, \quad \text{where} \quad s+p+q = 2n+1,$$

since the regular parts of ψ and φ contain only terms like the numerator of the last expression.

But the integral of such an expression multiplied by $\delta(\mathbf{x})$ is zero, because not all of the s, p, q are even and because $\delta(\mathbf{x})$ is spherically symmetric.

We shall see that all the expressions with which we shall deal in practice are of this type; that for them

$$\widetilde{\psi\varphi} = \tilde{\psi}\tilde{\varphi}. \tag{A.2.16}$$

This equality, to which we shall refer briefly as the tweedling of products, is assumed throughout this book. Later we shall see, in our concrete calculations, that this equality is indeed valid.

3. THE COVARIANT CHARACTER OF THE δ's. TENSORS ON WORLD-LINES

We ask: what are the transformation properties of an ordinary Dirac δfunction in four dimensions, defined by a generalization of the axioms (A_1-A_3) of Section 1 to this number of dimensions? The answer is simple enough. Obviously we must have

$$\int dx\, \delta_{(4)}(x) = 1. \tag{A.3.1}$$

This equality must have a covariant character. Since an integral is invariant only when the integrand is a scalar density, we deduce that $\delta_{(4)}$ is a scalar density, so that $\delta_{(4)}(-g)^{-1/2}$ is a scalar. (It would have been more consistent to use a different type for δ. We shall, however, refrain from doing so, keeping in mind that $\delta_{(4)}$ is a scalar density.)

We assume now that $\xi^\alpha(\lambda)$ is a world-line, such that through $\xi^0 = x^0 = \xi^0(\lambda)$ the parameter λ can be expressed as a function of x^0. Consider now a tensor $T^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q}(\lambda)$ defined only on such a world-line. This means: under an arbitrary coordinate transformation $x^\alpha = x^\alpha(x')$ these T's transform according to the formula:

$$T'^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q}(\lambda) = \left(\frac{\partial x'^{\alpha_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x'^{\alpha_p}}{\partial x^{\nu_p}} \cdot \frac{\partial x^{\mu_1}}{\partial x'^{\beta_1}} \cdots \frac{\partial x^{\mu_q}}{\partial x'^{\beta_q}}\right)\Bigg|_{x=\xi(\lambda)} T^{\nu_1 \cdots \nu_p}_{\mu_1 \cdots \mu_q}(\lambda). \tag{A.3.2}$$

We may apply the rules of tensor algebra, but not those of tensor analysis, to such tensors defined along the world-line. To be able to apply the latter, we must have tensor fields. We can, at least symbolically, change a tensor defined along a world-line into a tensor field, or rather into a tensor density field. This can be done in the following way:

$$\mathfrak{T}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q}(x) = \int_{-\infty}^{\infty} d\lambda\, \delta_{(4)}\big(x - \xi(\lambda)\big)\, T^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q}(\lambda). \tag{A.3.3}$$

Of course such a tensor density vanishes off the world-line and is singular on the world-line. Yet formally it can be treated as a tensor density field. Its transformation properties as a tensor density field can easily be verified from (A.3.2), from the fact that $\delta_{(4)}$ is a scalar density, and finally from the fact that (for continuous functions):

$$\delta_{(4)}(x - \xi) f(x) = \delta_{(4)}(x - \xi) f(\xi).$$

As a special application of this idea, we now introduce for $T^{\cdots}_{\cdots}(\lambda)$ a vector defined only along the curve, namely $d\xi^\alpha/d\lambda$. We can change this vector field symbolically into a vector density field:

$$\mathfrak{u}^\alpha = \int_{-\infty}^{\infty} d\lambda\, \delta_{(4)}(x - \xi) \frac{d\xi^\alpha}{d\lambda}. \tag{A.3.4}$$

We shall be especially interested in the zero component of this field:

$$\mathfrak{u}^0(x) = \int_{-\infty}^{\infty} d\xi^0\, \delta_{(4)}(x - \xi) = \delta_{(3)}\big(\mathbf{x} - \boldsymbol{\xi}(x^0)\big) = \delta(\mathbf{x} - \boldsymbol{\xi}). \tag{A.3.5}$$

This equation may be regarded as the definition of $\delta_{(3)}$ as far as its transformation properties are concerned; that is: it is the zero component of a vector density field.

We have introduced the $\delta_{(4)}$, which we shall meet only rarely in practical calculations, solely in order to find the transformation properties of $\delta_{(3)}$. Since the $\delta_{(3)}$ should be one of our good δ's we must redefine $\delta_{(4)}$, so that the good δ appears in (A.3.5). This is easy enough; the $\delta_{(4)}$ must satisfy the axioms (A_1-A_3) for a four-dimensional manifold; besides, it must satisfy (A.3.5) where the δ is a good δ. Thus (A.3.5) defines both the transformation properties of $\delta_{(3)}$ and the structure of $\delta_{(4)}$.

Although $\delta_{(3)}$ has rather complicated transformation properties, we do not need to be disturbed by this since it will appear (in practice) always in conjunction with $d\mathbf{x}$. Indeed, having a function $f(x^0, x^k)$, multiplying it by δ and integrating over a neighbourhood $V(\xi)$ of the point ξ on the world-line, at a certain moment x_0, we find, because of (A.3.5):

$$\int_{V(\xi)} f\delta(\mathbf{x}-\xi)\,d\mathbf{x} = \int_{V(\xi)}\int_{-\infty}^{\infty} f d\xi^0 \delta_{(4)}(x-\xi)\,d\mathbf{x} = f(\xi^0, \xi^n). \qquad (A.3.6)$$

Putting $f = 1$, we find:

$$\int \delta(\mathbf{x}-\xi)\,d\mathbf{x} = 1 \qquad (A.3.7)$$

which means that $\delta d\mathbf{x}$ can be treated as an invariant.

Since we have the connection (A.3.5) between $\delta_{(4)}$ and δ, we can write down in a different way the equations (A.3.3):

$$\begin{aligned}\mathfrak{T}^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}(x) &= \int_{-\infty}^{\infty} d\xi^0 \frac{d\lambda}{d\xi^0}\,\delta_{(4)}(x-\xi)\,T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\big(\lambda(\xi^0)\big)\\ &= \frac{d\lambda}{d\xi^0}\,T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\big(\lambda(\xi^0)\big)\Big|_{\xi^0=x^0}\int_{-\infty}^{\infty} d\xi^0\delta_{(4)}\big(x-\xi(\xi^0)\big)\\ &= \frac{d\lambda}{d\xi^0}\,T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\big(\lambda(x^0)\big)\,\delta\big(\mathbf{x}-\xi(x^0)\big). \end{aligned} \qquad (A.3.8)$$

The last equation shows that the singularity of the symbolic tensor density field $\mathfrak{T}$ is a singularity of the δ type along the world-line. One can deduce from it the opposite formula expressing a tensor along a world-line in terms of the tensor density field:

$$T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}(\lambda) = \frac{d\xi^\varrho}{d\lambda} \int_\sigma d\sigma\, n_\varrho \mathfrak{T}^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}(x) \tag{A.3.9}$$

where σ is a hypersurface on which the point $x^\alpha = \xi^\alpha(\lambda)$ lies.

These symbolic fields, defined by (A.3.3), will play an essential role in our analysis. Such fields conserve formally the usual properties of the ordinary tensor densities, with respect to parallel displacement and covariant differentiation.

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